Conjugacy and reciprocal bases in anisotropic dielectric media

Conjugación y bases recíprocas en medios dieléctricos anisótropos

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ABSTRACT:
The intrinsic geometrical structure associated with propagation of plane waves in anisotropic dielectric media is analyzed. In addition to allow the change of variance in components of electric magnitudes, the use of reciprocal vector bases reveals some relationships in their propagation analogous to crystal structures. As an application, the case when the direction of propagation $n$ lies in a principal plane is discussed showing the analogy between this subject and a problem of elastic bending of homogeneous beams. This analogy allows mutual transfer of results between these disciplines. Among them, the use in optics of a trivial graphical construction to find conjugate directions in a plane problem.

Keywords: Dielectric Anisotropy, Conjugate Directions, Reciprocal Bases, Elasto-Optical Analogies.

RESUMEN:
Se analiza la estructura geométrica intrínseca asociada a la propagación de ondas planas en medios dieléctricos anisótropos. El uso de bases vectoriales recíprocas, además de permitir el cambio de varianza en las componentes de las magnitudes eléctricas, nos descubre relaciones análogas a la estructura cristalina en la propagación de las mismas. Como aplicación práctica, se considera el caso en que la dirección de propagación $n$ se encuentra en un plano principal, mostrando la analogía con la flexión elástica de vigas homogéneas en Elasticidad. Esta analogía permite una transferencia mutua de resultados entre las dos disciplinas, como la utilización en óptica de una construcción gráfica trivial para encontrar direcciones conjugadas en un problema plano.

Palabras clave: Anisotropía Dieléctrica, Direcciones Conjugadas, Bases Recíprocas, Analogías Elasto-Ópticas.

REFERENCIAS Y ENLACES / REFERENCES AND LINKS
1. Introduction

Optics of anisotropic materials (like crystals) has always deserved great interest, albeit its complexity. At present, the study of these media is relevant when dealing with liquid [1,2] and photonic [3] crystals and several kind of metamaterials [4].

The main aim of this work is to analyse the intrinsic geometrical structure associated with propagation of light plane waves in anisotropic dielectric media, that seems unnoticed in the literature. Reciprocal vector bases has been introduced together with their associated coordinates and the analogy with crystal structures (real and reciprocal space) has been shown.

First, the general plane wave equation is derived following two alternative procedures. In the former, double (both time and position) Fourier transform of Maxwell equations has been carried out and in the latter, jump conditions for electromagnetic fields at a wavefront are applied by an observer travelling with the wave-front. With respect to this wave equation, some corollaries are derived. These corollaries show that if direction of propagation \( \mathbf{n} \) and electric fields \( \mathbf{E}_1 \) and \( \mathbf{E}_2 \) lie along conjugate directions, general plane wave equation is satisfied (necessary condition in lossless plane wave propagation) and, reciprocally, if this equation is verified, directions \( \mathbf{n} \), \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) are conjugate (sufficient condition). The use of non-orthogonal curvilinear coordinates as a practical tool is also highlighted. Expressions for Poynting vectors in terms of covariant and contravariant vectors are found and the procedure is applied to the case of wave propagation in a principal plane. In Section 5, a simple graphical construction based on opto-elastical analogies is presented and applied to the mentioned case.

Linear media, with dielectric anisotropy (tensor \( \varepsilon \)), magnetic isotropy \( (\mu = \mu_0 \mu_r) \) and free of currents and charges \( (j = 0; \rho = 0) \) are considered. The constitutive (or material) relation between the displacement vector \( \mathbf{D} \) and the electric field vector \( \mathbf{E} \) can be expressed as \( \mathbf{D} = \varepsilon_0 \varepsilon \mathbf{E} \), where \( \varepsilon_0 \) is the permittivity of vacuum and \( \varepsilon \) is the relative dielectric permittivity tensor.

The general plane wave equation in anisotropic dielectric media can be expressed as:

\[ k^2 E - (k \cdot E) k = \mu \omega^2 D, \]  

(1)

where \( k \) is the wave vector whose length is equal to \( \omega / v_p \). Parameters \( \omega \) and \( v_p \) are the angular frequency and phase velocity, respectively. In the next section, it is shown how two different electromagnetic processes (propagation of disturbances through the medium and application of boundary conditions at the wavefront) are represented by the same equation (Eq. (1)). It must be noted that the use of Fourier transform maps differential or integro-differential equations from its original “domain” (differential equations from its original electromagnetic processes) to Maxwell equations) for these media:

Our starting point is the set of equations of evolution (Maxwell equations) for these media:

\[
\begin{align*}
\nabla \cdot D &= 0, \\
\nabla \times D &= -\frac{\partial B}{\partial t}, \\
\nabla \cdot B &= 0, \\
\n\nabla \times H &= -\frac{\partial D}{\partial t}.
\end{align*}
\]

(2) - (5)

The double Fourier transform of a vector field \( F(t, r) \), function of both time and position is given by [5]:

\[
\tilde{F}(\omega, k) = \int dt dr e^{i(\omega t - k \cdot r)} F(t, r),
\]

(6)

and, consequently:

\[
\frac{\partial F}{\partial t} \rightarrow -i \omega \tilde{F}(\omega, k), \quad \nabla \cdot F(t, r) \rightarrow i k \cdot \tilde{F}(\omega, k),
\]

\[
\nabla \times F(t, r) \rightarrow i k \times \tilde{F}(\omega, k).
\]

The Fourier transformed form of the four Eqs. (2) is then:

\[
\begin{align*}
k \cdot D &= 0, \\
k \times E &= \omega B, \\
k \cdot B &= 0, \\
k \times H &= -\omega D.
\end{align*}
\]

(7) - (10)

From Eq. (10) and since \( B = \mu_0 \mu_r H \), vector product of \( k \) and Eq. (8) gives:

\[
k \times (k \times E) = \mu_0 \mu_r \omega k \times H = -\mu_0 \mu_r \omega^2 D,
\]

(11)

and Eq. (1) is reobtained.

1.2. The jump conditions for a Galilean observer situated at the wave front [6]

According to Galilean transformation formulae [7] for \( E, D, H \) and \( B \) connecting any two inertial frames, one has:

\[
E' = E + v \times B, \quad B' = B, \\
H' = H - v \times D, \quad D' = D,
\]

(12) - (13)

where \( v \) is the constant velocity of the primed with respect to the unprimed frame. In this procedure, the wavefront is considered a moving discontinuity surface. Jump conditions are applied by an observer situated at the wavefront and travelling with it. For the moving observer this is a "static" problem of boundary conditions. If \( n \) is the unit vector normal to the wavefront, one can write the four jump conditions:

\[
\begin{align*}
n \times [E'] &= 0, \\
n \cdot [B'] &= 0, \\
n \times [H'] &= 0, \\
n \cdot [D'] &= 0,
\end{align*}
\]

(14) - (15)

where \( [\ ] \) denotes the "jump" in vector field. From the first Eq. of (14), one has:

\[
n \times (0 - E') = 0 \Rightarrow n \times E + n \times (v \times B) = 0,
\]

(16)

because the observer considers that there is not any field in points that are not yet reached by the disturbance. But, taking into account the second Eq. of (12), and that \( n \times (0 - B') = 0 \), one has:

\[
n \times (v \times B) = -(n \cdot v)B.
\]

(17)

The vector product of \( n \) and Eq. (16) gives:

\[
n \times (n \times E) = n \times B(n \cdot v).
\]

(18)

And, from the first equation of Eq. (15), one obtains \( n \times (H - v \times E) = 0 \). Since \( n = k/k \), \( B = \mu_0 \mu_r H \), and \( n \cdot D = 0 \):

\[
k \times (k \times E) = (k \cdot E)k - k^2 E = -\mu_0 \mu_r (k \cdot v)^2.
\]

(19)

and Eq. (1) is obtained, because phase velocity is given as \( v_p = v \cdot n \), and \( k \cdot v = \omega \).
The main result is that both alternative derivations lead to the same equation.

2. Conjugate directions: Terminology and notation

Let \( \vec{I} \) be a cartesian symmetric tensor of second order, with components \((I_{ij})\) referred to a cartesian reference frame and let \( \hat{u} \) and \( \hat{v} \) be unit vectors. The bound vector to \( \hat{u} \) by the tensor \[8\] is vector \( I_u \) defined as

\[ I_u = \vec{I} \cdot \hat{u}. \]

The intrinsic component, \( \sigma_u \) of vector \( I_u \) is equal to its projection onto vector \( \hat{u} \), in such a way that

\[ \sigma_u = I_u \cdot \hat{u}. \]

From Cauchy’s theorem \[9\], one has that

\[ I_u \cdot \hat{v} = I_v \cdot \hat{u}. \]

**Definition:** According to our notation, directions \( \hat{u} \) and \( \hat{v} \) are conjugate \[10,11\] if:

\[ I_u \cdot \hat{v} = I_v \cdot \hat{u} = 0. \] (20)

Equation (20) provides a simple geometrical interpretation: An important property of the Cauchy’s quadric \[12\] is that its normal at every point is collinear with bound vector \( I_u \) for any direction \( \hat{u} \) (see Fig. 1).

The orthogonality of vectors \( I_u \) and \( \hat{v} \) (and, from Cauchy theorem, \( I_v \) and \( \hat{u} \)) implies that \( \hat{u} \) y \( \hat{v} \) are conjugate directions. Figure 2 shows the intersection of ellipsoid and a coordinate plane (by instance, \( x_3=0 \)). The slopes of conjugate directions, \( \phi \) and \( \psi \), verify:

\[ \tan \psi \tan \phi = \frac{I_1}{I_2} \] (21)

where \( I_1 \) e \( I_2 \) are the eigenvalues of the two-dimensional tensor.

3. Conjugate directions in anisotropic media

Consider plane light waves in anisotropic dielectric media with magnetic isotropy. An Euclidean frame along the principal directions of tensor \( \vec{E} \) is used. It is assumed that \( \varepsilon_1 > \varepsilon_2 > \varepsilon_3 \) are the eigenvalues of tensor \( \vec{E} \), that can be positive or negative (the used methodology is valid for both positive and negative values of \( \varepsilon_i \) and \( \mu_r \) allowing the study of left-handed (LH materials). Furthermore, dispersion is disregarded and eigenvalues \( \varepsilon_i \) are the constants depending only on the material.

In this section, it will be shown that the direction of propagation \( \hat{n} = \vec{k}/|\vec{k}| \) and the directions of the electric fields \( \vec{E}_1 \) and \( \vec{E}_2 \) of the two unique eigenmodes associated with \( \hat{n} \) are conjugate directions.

Taking the dot product of Eq. (1) and vector \( \vec{D} \), one has that:

\[ \nu_p^2 = \frac{2W_e}{\mu |\vec{D}|^2} \] (22)
where $W_e = E \cdot D / 2$ is the electric energy density and therefore values of $v_p$ are inversely proportional to $|D|$. 

The general equation (1) can be written in an alternative way [13-15] as:

$$\dot{v} - (\hat{n} \cdot \dot{v})\hat{n} = \frac{v_p^2}{c^2} \varepsilon_v$$

(23)

where unit vector $\dot{v}$ is defined as $\dot{v} = E / |E|$, bound vector $\varepsilon_v = \varepsilon_0 D / |E|$, and parameter $a$ is defined as $a = \mu_r v_p^2 / c^2$. 

In connection with Eq. (1), some corollaries are derived by the authors. Other corollaries shown in textbooks are briefly summarized with corresponding references.

**Corollary 1.** For lossless media and assuming that vectors $\dot{v}$ and $\hat{n}$ are conjugate, the necessary condition for attaining extrema of phase velocities is that Eq. (1) must be satisfied.

In other words, it is intended to evaluate extrema of $|D|^2$ (inverse of those of $v_p^2$) subject to the following constraints:

- The electric energy density is a constant:

$$W_e = \frac{1}{2} E \cdot D = cte$$

(24)

- Vectors $E$ and $k$ are along conjugate directions, and then:

$$D \cdot k = 0 \quad \text{with} \quad D = \varepsilon_0 \varepsilon \cdot E$$

(25)

**Proof:** Extrema of $|D|^2$ can be found by constructing function $\phi$ and using Lagrange’s multipliers $\lambda$ and $\gamma$:

$$\delta \phi = \delta \left( |D|^2 + \lambda (E \cdot D - W_e) + \gamma (D \cdot k) \right) = 0.$$

The necessary condition for the extremum leads to the verification of the following relations:

$$2D + \lambda E + \gamma k = 0,$$

(26)

with $W_e = \frac{1}{2} E \cdot D$ and $D \cdot k = 0$, that enables us to obtain $\lambda$ and $\gamma$:

$$\lambda = -\frac{|D|^2}{W_e}, \quad \gamma = \frac{|D|^2 (E \cdot k)}{W_e k^2}.$$

Substitution of these parameters in Eq. (26) leads to (1).

Verification of (1) implies the following corollaries (sufficient conditions).

**Corollary 2.** Vectors $E$, $D$ and $\hat{n}$ are coplanar (It is obvious because of the linear dependence).

**Corollary 3.** Directions of $\hat{n}$ and $\dot{v} = E / |E|$ are conjugate.

**Proof:** If we multiply both sides of Eq. (23) scalarly by $\hat{n}$, since $\mu_r v_p^2 / c^2 \neq 0$, we obtain:

$$\dot{v} \cdot \hat{n} - (\hat{n} \cdot \dot{v})\hat{n}^2 = \frac{v_p^2}{c^2} \varepsilon_v \cdot \hat{n} \Rightarrow \varepsilon_v \cdot \hat{n} = 0,$$

Equation (27) shows that the directions of $\hat{n}$ and $\dot{v}$ are conjugate.

**Corollary 4.** There exist only two nontrivial solutions of the phase velocity, $v_p$ compatible with the general plane wave equation. (See [16])

For every $\hat{n}$, there are two solutions $a_1$ and $a_2$, where $a = \mu_r v_p^2 / c^2$, given by the expression [13]:

$$a_{1,2} = \frac{i \sigma_n - \varepsilon_n^2 \pm \sqrt{(i \sigma_n - \varepsilon_n^2)^2 - 4 \sigma_n \Delta}}{2 \Delta},$$

(28)

where $\varepsilon_n = \varepsilon_0 \varepsilon$, $\sigma_n = \varepsilon_0 \varepsilon \sigma$, $I = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$ the trace of tensor $\varepsilon$ and $\Delta = \varepsilon_1 \varepsilon_2 \varepsilon_3$ its determinant. Therefore, there exist two unique values of $\dot{v}$, (say, $\dot{v}_1$ and $\dot{v}_2$) that verify Eq. (1). Thus, $\dot{v}_1$ and $\dot{v}_2$ are conjugate directions with $\hat{n}$, according to Corollary 3.

**Corollary 5.** Directions $\dot{v}_1 = E_1 / |E_1|$ and $\dot{v}_2 = E_2 / |E_2|$ are conjugate.

**Proof:** If Eq. (23) is particularized for $\dot{v}_1$ and $\dot{v}_2$, we can write:

$$\dot{v}_1 - (\hat{n} \cdot \dot{v}_1)\hat{n} = a_1 \varepsilon_v,$$

(29)

$$\dot{v}_2 - (\hat{n} \cdot \dot{v}_2)\hat{n} = a_2 \varepsilon_v,$$

(30)

where $a_1 = \mu_r v_{p1}^2 / c^2$ and $a_2 = \mu_r v_{p2}^2 / c^2$, and it is assumed that $v_{p1} \neq v_{p2}$. By multiplying Eq. (29) by $\dot{v}_2$ and Eq. (30) by $\dot{v}_1$ and subtracting, one has that:
\[ \varepsilon_n \hat{n} = \varepsilon_{n\prime} \hat{n} = 0, \]  
(31)
where Cauchy’s theorem is taken into account.

**Corollary 6.** Vectors \( \mathbf{D}_1 \) and \( \mathbf{D}_2 \) are orthogonal. (See [16]).

**Corollary 7.** The bound vector \( \varepsilon_n \) is orthogonal to both \( \hat{n} \) and \( \hat{n}\prime \).

**Proof:** From Corollary 3 and applying Cauchy’s theorem, one has that:

\[ \varepsilon_n \cdot \hat{n} = 0; \quad \varepsilon_n \cdot \hat{n}\prime = 0, \]
(32)
and, consequently, the bound vector to \( \varepsilon_n \) is orthogonal to the plane defined by the fields \( \mathbf{E}_1 \) and \( \mathbf{E}_2 \).

To summarize, vector \( \mathbf{D}_1 \) is orthogonal to \( \hat{n} \) and \( \hat{n}\prime \). Moreover, vector \( \mathbf{D}_2 \) is orthogonal to \( \hat{n} \) and \( \hat{n}\prime \). The plane determined by vectors \( \hat{n} \), \( \mathbf{D}_1 \) and \( \hat{n}\prime \) and the plane determined by \( \hat{n}\prime \), \( \mathbf{D}_2 \) and \( \hat{n} \) are orthogonal, with \( \hat{n} \) belonging simultaneously to both planes (See Fig. 3).

![Fig. 3: Directions of the wave normal \( \hat{n} \) and the field vectors of the two eigenmodes associated with it in dielectric anisotropic media. Vectors \( \hat{n} \), \( \hat{n}\prime \) and \( \hat{n} \) are conjugate directions. The bound vector to \( \varepsilon_n \), \( \varepsilon_{n\prime} \) is orthogonal to the plane formed by \( \mathbf{E}_1 \) and \( \mathbf{E}_2 \).](image)

4. **Conjugacy and reciprocity: Direct and reciprocal bases**

In every Maxwell equation, as in every well formulated law of physics, both members of the equation have the same tensor nature and this intrinsical character makes them valid in every coordinate frame. Nevertheless, if their formulation is restricted to orthogonal frames, in which there is no distinction between covariant and contravariant vector components, the different tensor nature of vector fields \( \mathbf{E} \) and \( \mathbf{D} \) remains masked (the same for \( \mathbf{B} \) and \( \mathbf{H} \)).

In this paper, the Euclidean space is endowed, at every point \( P \), with two reciprocal bases defined from physical properties of the medium (Cauchy quadric). The fundamental metric tensor \( (g_{ij}) \) and its reciprocal one \( (g^{ij}) \) are determined and, consequently, transformations for changing the variance of tensor components are proposed.

It is well known that for general curvilinear coordinates \( (q_1, q_2, q_3) \), defined at a point \( P \) in three-dimensional Euclidean space, direct basis vectors \( \mathbf{e}_i \) (that lie along coordinate lines) and reciprocal basis vectors \( \mathbf{e}^j \) (normal to the coordinate surfaces) are defined as: [17]:

\[ \mathbf{e}_j = \frac{\partial \mathbf{r}}{\partial q_j} = \sum_{k=1}^{3} \frac{\partial x_k}{\partial q_j} \mathbf{u}_k, \quad j \in (1,2,3), \]
(33)
\[ \mathbf{e}^j = \nabla q_j = \sum_{k=1}^{3} \frac{\partial q_j}{\partial x_k} \mathbf{u}_k, \quad j \in (1,2,3), \]
(34)
where \( (\partial, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) \) is a set of unit vectors of a cartesian frame. In this case, these vectors lie along principal directions of the dielectric permittivity tensor and verify that:

\[ \mathbf{e}_i \cdot \mathbf{e}^j = \delta^j_i, \]
(35)
where \( \delta^j_i \) is the Kronecker delta.

We have shown that directions \( \mathbf{n}, \mathbf{v}_1, \mathbf{v}_2 \) are mutually conjugate. So, we can construct two local bases of vectors \( (P, \mathbf{e}_i) \) and \( (P, \mathbf{e}^j) \) with \( i \in (1,2,3) \) given by:

\[ \mathbf{e}_1 = \mathbf{v}_1, \quad \mathbf{e}_2 = \mathbf{v}_2, \quad \mathbf{e}_3 = \hat{n}, \]
\[ \mathbf{e}^1 = \frac{\varepsilon_{v_1}}{\sigma_1}, \quad \mathbf{e}^2 = \frac{\varepsilon_{v_2}}{\sigma_2}, \quad \mathbf{e}^3 = \frac{\varepsilon_n}{\sigma_3}, \]
(36)
where \( \sigma_1 = \sigma_{v_1} = \varepsilon_{v_1} \cdot \mathbf{v}_1, \quad \sigma_2 = \sigma_{v_2} = \varepsilon_{v_2} \cdot \mathbf{v}_2, \quad \sigma_3 = \sigma_n = \varepsilon_n \cdot \mathbf{n} \). Vectors of these bases verify Eq. (35), and then the bases are reciprocal (Fig. 4).

In Table I it is shown that electromagnetic fields have an only component. This means that the wave front carries a geometrical structure that remains the reciprocal bases of crystals.

According to definition (Eq. (36)), if vectors of the direct base \( (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \) have the following components in this cartesian frame of principal directions: \( \mathbf{e}_1 = (\alpha_1, \alpha_2, \alpha_3), \quad \mathbf{e}_2 = (\beta_1, \beta_2, \beta_3), \)
Tabla I  
Vectores de bases directas y recíprocas

<table>
<thead>
<tr>
<th>Bases directas</th>
<th>Bases recíprocas</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_1$, colineal con $D_1$ y $H_2$</td>
<td>$e_1^\ast$, colineal con $D_1$ y $H_2$</td>
</tr>
<tr>
<td>$e_2$, colineal con $D_2$ y $H_1$</td>
<td>$e_2^\ast$, colineal con $D_2$ y $H_1$</td>
</tr>
<tr>
<td>$e_3 \equiv n$</td>
<td>$e_3^\ast$, colineal con $\varepsilon_n$</td>
</tr>
</tbody>
</table>

Fig. 4: Coordenadas curvilíneas y bases directas e inversas asociadas con un plano de onda en dielectroide anisotrópico.

$
\mathbf{e}_3 = (y_1, y_2, y_3),
$
e pueden definir superficies coordenadas por medio de una transformación lineal

$q_i = q_i(x_1, x_2, x_3),
$
de $dq_i = \mathbf{e}_i \cdot d\mathbf{r}$ y, de acuerdo con un primer enfoque, uno puede escribir:

$\begin{pmatrix}
q_1 \\
q_2 \\
q_3
\end{pmatrix} =
\begin{pmatrix}
\varepsilon_1 \alpha_1 & \varepsilon_2 \alpha_2 & \varepsilon_3 \alpha_3 \\
\varepsilon_1 \beta_1 & \varepsilon_2 \beta_2 & \varepsilon_3 \beta_3 \\
\sigma_1 & \sigma_2 & \sigma_3
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix},

(37)

Esta transformación es invertible y uno debe notar que $q_i = \text{const}$ es la ecuación de una superficie coordenada que, en el presente caso, reduce a planos tangentes a las superficies coordenadas en $P$.

### 4.1. Tensor métrico fundamental

Tomando en cuenta la definición de los vectores de bases, Eqs. (29) y (30) se pueden escribir como:

$a_1 \sigma_1 \mathbf{e}_1 = \mathbf{e}_1 - g_{12} \mathbf{e}_3,

(38)

$a_2 \sigma_2 \mathbf{e}_2 = \mathbf{e}_2 - g_{23} \mathbf{e}_3,

(39)$

donde $g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$ denotan los elementos de la matriz de Gram. Dado que $\mathbf{e}_i = a_i \sigma_i \mathbf{e}_i$ se pueden referir ambos miembros de la Ecuación (38) a la misma base:

$a_1 \sigma_1 \mathbf{e}_1 = \sum_{j=1}^{3} (g_{ij} - g_{13} g_{3j}) \mathbf{e}_j,

(40)

j \in (1,2,3),

$a_2 \sigma_2 \mathbf{e}_2 = \sum_{j=1}^{3} (g_{2j} - g_{23} g_{3j}) \mathbf{e}_j,

(41)

j \in (1,2,3).$

Trazado de la matriz $g_{ij}$

La matriz $g_{ij}$ puede obtenerse de la ecuación precedente:

$\begin{pmatrix}
s_1 s_2 s_1 \\
1 s_2 \\
s_1 s_2
\end{pmatrix}

(42)$

donde $s_1 = \sqrt{1 - a_1 \sigma_1}, s_2 = \sqrt{1 - a_2 \sigma_2}$, y el signo positivo de las raíces es considerado. Refiriendo Eq. (38) a la base en donde se consideran componentes covariantes se tiene:

$a_1 \sigma_1 g_{ij} \mathbf{e}_i = \mathbf{e}_i - g_{13} \mathbf{e}_3$, para $i = 1,2.$

Sumando para $j$, tomando $i = 1$ y $i = 2$ y identificando términos:

$a_1 \sigma_1 (g_{11} \mathbf{e}_1 + g_{12} \mathbf{e}_2 + g_{13} \mathbf{e}_3) = \mathbf{e}_1 - g_{13} \mathbf{e}_3,

a_2 \sigma_2 (g_{21} \mathbf{e}_1 + g_{22} \mathbf{e}_2 + g_{23} \mathbf{e}_3) = \mathbf{e}_2 - g_{23} \mathbf{e}_3,

$el inverso de la matriz $g_{ij}$ se obtiene:

$\begin{pmatrix}
1 & 0 & -s_1 \\
0 & 1 & -s_2 \\
-s_1 & -s_2 & 1
\end{pmatrix}

(43)$

and, since $[g_{ij}][g^{ij}] = [\delta_{ji}^j]$ the Jacobian of the transformation is:

$\mathbf{J} = \sqrt{|G|} = a_1 a_2 \sigma_1 \sigma_2.$

El coeficiente $g^{33}$ se obtiene a partir de $g^{33} = G^{33}/G$, donde $G^{33}$ es el menor de $g_{33}$ y se define:

$g^{33} = \frac{|\varepsilon_n|^2}{\sigma_2^2} = \frac{1}{a_1 \sigma_1} - \frac{1}{a_2 \sigma_2} - 1.

(44)$

Matriz $[g_{ij}]$ (Eq. (43)) se convierte en la matriz unitaria, a excepción de $a_1 \sigma_1 = a_2 \sigma_2 = 1$. Matrices $\{g_{ij}\}$ y $\{g^{ij}\}$ son matrices de Gram, lo que permite el cambio de bases.
4.2. Bound vector $\epsilon_n$ and Poynting vector

In anisotropic media, to every direction $n$, there correspond two modes of propagation, say 1 and 2, and an only vector $\epsilon_n$. Thus, for mode 1, vector $S_1$ in the direction of ray 1, is written in direct and reciprocal bases as:

$$S_1 = \lambda e_1 \wedge e^2 = \frac{1}{\sqrt{1-s_1^2\sigma^2}}(e^3 + s_2 e^2) = -\frac{\eta}{\sqrt{|G|}}(e_3 - s_1 e_1).$$  \hspace{1cm} (45)

where $\lambda$ and $\eta$ are constants. Similarly, a vector $S_2$ in the direction of the ray 2 is given by:

$$S_2 = \lambda' e_2 \wedge e^1 = -\frac{\eta'}{\sqrt{|G|}}(s_2 e_2 - e_3).$$ \hspace{1cm} (46)

where $|G| = a_1a_2\sigma_2$ and $\lambda'$ and $\eta'$ are constants.

In uniaxial media, one can easily show that vector $\epsilon_n$ is in the direction of the extraordinary ray. If modes 1 and 2 denote extraordinary and ordinary modes, respectively, one has that $e_2$ is parallel to $e^2$ and $e_3 = 1/a_2$. Hence,

$$S_1 = \lambda e_1 \wedge e_2 = \frac{1}{\sqrt{|G|}}|e^3|,$$

$$S_2 = \lambda e_1 \wedge e_2 = -\frac{\eta'}{\sqrt{|G|}}e_3.$$

In uniaxial media (positive or negative), the extraordinary ray is parallel to $e^3$. The ordinary ray is in the direction of $e_3$.

In general, in biaxial media, rays $S_1$ and $S_2$ are not in the direction of $e^3$ nor $e_3$, as Eqs. (45) and (46) show, unless $a_i\sigma_i = 1$, for $i=1$ or 2. This is the case when vector $\tilde{n}$ lies in a principal plane, as it is shown in the application of the next subsection.

4.3. Application of the procedure

Consider light propagation in biaxial media when vector $\tilde{n}$ lies in a principal plane. Without loss of generality, let us assume that $\tilde{n} = (n_1, n_2, 0)$. In this case, from Eq. (28), one has that $a_1 = 1/\epsilon_3$, $a_2 = \sigma_n/\epsilon_2\epsilon_3$, and vectors $e_1$ and $e^1$ can be written in the orthogonal base of principal axes in the form:

$$e_1 = \left(\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right), e_2 = \frac{1}{|e_n|}\left(\begin{array}{c} -\epsilon_2 y_2 \\ \epsilon_1 y_1 \\ 0 \end{array}\right), e_3 = \left(\begin{array}{c} y_1 \\ 0 \\ 0 \end{array}\right).$$ \hspace{1cm} (47)

$$e^1 = e_1, e^2 = \frac{|e_n|}{\sigma_n}\left(\begin{array}{c} -y_2 \\ y_1 \\ 0 \end{array}\right), e^3 = \frac{1}{\sigma_n}\left(\begin{array}{c} \epsilon_1 y_1 \\ 0 \\ 0 \end{array}\right).$$ \hspace{1cm} (48)

In this case, mode 1 (for $a_1 = 1/\epsilon_3$) is the “ordinary” mode, and the corresponding ray is collinear with wave vector $(e_3)$. For the other mode, there is an “extraordinary” ray in the direction of $e^3$. Figure 5 shows these vectors together with $e_2$ (collinear with $E_2$) and $e^2$ (collinear with $D_2$). These four vectors lie in the plane $x_3=0$. Note that $e^1 \equiv e_1$ is in the $x_3$ direction ($E_1$ is collinear with $D_1$). Conjugation of vectors $e_3$ and $e^2$ is shown in the figure and simple calculations lead to the expression relating their slopes (see Eq. (21):

$$\tan \phi = \frac{\epsilon_3}{\epsilon_2}.$$ \hspace{1cm} (49)

Since $\tilde{E} = \epsilon_{ij} e_i \otimes e_j$ and according to tensor property of contraction multiplication, covariant components of permittivity tensor can be easily obtained as:

$$\epsilon_{ij} = e_i \cdot \tilde{E} \cdot e_j.$$
In particular, for the present case, the dielectric permittivity tensor reduces to:

\[
\varepsilon_{ij} = \begin{pmatrix}
\sigma_1 & 0 & 0 \\
0 & \sigma_2 & 0 \\
0 & 0 & \sigma_3 \\
\end{pmatrix},
\]

and its associated quadric becomes:

\[
\sigma_1 q_1^2 + \sigma_2 q_2^2 + \sigma_3 q_3^2 = 1.
\]

The plane normal to vector \( e^1 \) (vector parallel to \( D_1 \), an consequently, parallel to \( H_2 \)) passing through the origin is given by \( q_1 = 0 \). This plane is spanned by vectors \( e_2 \) and \( e_3 \), as it is shown in Fig. (5). For \( q_2 = 0 \), one has the direction of vector \( e_3 \) and, for \( q_3 = 0 \), one gets the direction of \( e_2 \).

The intersection of ray ellipsoid with the plane \( q_1 = 0 \) gives the ellipse:

\[
\sigma_2 q_2^2 + \sigma_3 q_3^2 = 1.
\]

The gradient of a function \( f(q_1,q_2,q_3) \) in curvilinear coordinates is given by:

\[
\nabla f = e^i \frac{\partial f}{\partial q_i}
\]

For \( f = \sigma_2 q_2^2 + \sigma_3 q_3^2 - 1 \):

\[
\nabla f = 2q_2 \sigma_2 e^2 + q_3 \sigma_3 e^3.
\]

For \( q_2 = 0 \), the gradient has the expression:

\[
\nabla f = 2q_3 \sigma_3 e^3
\]

and is in the direction of the extraordinary ray. This can be visualized too in Fig. (5).

5. The elastic analogue

Conjugate directions are widely used in Strength of materials. When dealing with elastic bending of homogeneous beams under an unsymmetrical loading [18,19], the neutral axis and the loading axis are conjugate with respect to the centroidal inertia tensor of the beam cross section. The neutral axis is the locus of points in the cross section, where the bending stress and strain vanish. The loading axis is the intersection between the loading plane and the cross section. The bending moment \( (M) \) is orthogonal to the loading axis.

There is a clear analogy between this problem of Strength of materials and the above case of optics of anisotropic media and a correspondence between optical and mechanical magnitudes can be stated: bending moment \( M \) is the analogue of \( D \) and the neutral axis is the analogue of \( E \) direction. Therefore, we can borrow some graphical constructions from strength of materials to be used in optics, for instance those based on Land’s circle [19].

For that reason, it is trivial to find the direction of the electric field \( E_2 \) graphically from the direction of \( \hat{n} \). As an application, let us consider again the case shown in previous subsection 4.3 (see Fig. 6). In the rectangular frame \( Ox_1x_2 \), segments \( OA = e_1 \) and \( AB = e_2 \) are taken on the axis \( Ox_2 \). On \( OB \) as diameter, we draw a circumference centred at point \( C \), of coordinates \( (O, (e_1 + e_2)/2) \) of radius \( R = e_1 + e_2/2 \) and tangent at \( O \) to the axis \( Ox_1 \).

If vector \( k \) forms an angle \( \phi \) with the axis \( Ox_1 \), its direction intersects the circumference at point \( P \). The straight line passing through \( P \) and \( A \) pierces the circle at point \( Q \). Then, \( QQ \) is the direction of vector \( E_2 \). From the drawing, it is easy to obtain Eq. (49) again: \( \tan \phi \tan \psi = e_1/e_2 \). Moreover, this construction is reciprocal, so we can find the direction of \( \hat{n} \) if direction of \( E_2 \) is known.

Fig. 6: Application of Lands circle (a graphical procedure borrowed from Elasticity) to find vector field \( E_2 \) when the direction of propagation lies in a principal plane of an anisotropic biaxial medium. If vector \( k \) forms an angle \( \phi \) with the axis \( Ox_1 \), its direction intersects the circumference at point \( P \). The straight line passing through \( P \) and \( A \) pierces the circle at point \( Q \). Then, \( QQ \) is the direction of vector \( E_2 \).
This procedure holds for any kind of anisotropic media, for instance, an interesting and valuable kind of metamaterials: the recently designed indefinite anisotropic media [4,20].

6. Conclusions

Some unnoticed properties dealing with conjugacy of vectors and direct and reciprocal bases are stated in the realm of optics of anisotropic dielectric media. Since propagation of electromagnetic waves are always described in orthogonal frames, where there is no distinction between covariant and contravariant vector components, the different tensor nature of vector fields \(E\) and \(D\) remains masked (the same for \(B\) and \(H\)). It is shown that an intrinsic geometrical structure is associated with propagation of plane waves in these media in such a way that the Euclidean space is endowed, at every point \(P\), with two reciprocal bases defined from physical properties of the medium.

First of all, we have also proposed alternative derivations of the general plane wave equation and its range of validity is discussed. Once the concept of conjugacy of vectors is introduced with its own terminology and notation, it is shown that the wave vector \(k\) and vectors \(E_1\) and \(E_2\) of the two unique eigenmodes associated with \(n\) are mutually conjugate. A generally non-orthogonal covariant basis formed by these conjugate unit vectors \((e_1 = E_1/|E_1|, e_2 = E_2/|E_2|\) and \(e_3 = k/|k|\)) is introduced. Another contravariant base related to bound vectors to the above vectors: \((e^1, e^2, e^3)\) is also built. Both bases behave as direct and reciprocal ones and remain of the geometry of crystal structures.

Adaptive curvilinear coordinates associated with these bases are also introduced. It is highlighted the biunivocal correspondence between the general plane wave equation and conjugacy of vectors and some new corollaries from this equation are derived. The fundamental metric tensor \(g_{ij}\) and its reciprocal one \(g^{ij}\) are determined and, consequently, transformations for changing the variance of tensor components are proposed. Properties of the ray ellipsoid (Cauchy quadric associated with dielectric permittivity tensor), dealing with this description are also investigated.

The relation between Poynting vector and vectors of direct and reciprocal bases is shown. It is found that, in uniaxial media, Poynting vector is in the direction of \(e_3\) or \(e^3\). This is also the case of propagation in biaxial media, when the direction of propagation lies in a principal plane: a detailed discussion can be found in Section 4. Finally, the analogy with a problem of elastic bending of homogeneous beams under an unsymmetrical loading is pointed out (Section 5). This analogy allows the use in optics of a trivial graphical construction to find conjugate directions in a plane problem.

Moreover, the used methodology is valid for both positive and negative values of dielectric permittivities and magnetic permeability and allows the study of light propagation in both right-handed (RH) and left-handed (LH) materials like, for instance, the recently designed indefinite dielectric media where dielectric permittivities are not all the same sign.