A METHOD TO INTEGRATE ANY CONTINUOUS FUNCTION OVER THE CHROMATICITY DIAGRAM

J. M. Zoido[♮], *F. Carreño*[♮] and *E. Bernabeu*[♯]
[♮] Escuela Universitaria de Óptica
Universidad Complutense de Madrid
C/ Arcos de Jalón s/n 28037 Madrid, Spain
E-Mail: fiopt11@emducms1.sis.ucm.es
[♯] Departamento de Óptica
Facultad de Ciencias Físicas
Universidad Complutense de Madrid
Ciudad Universitaria s/n 28040 Madrid, Spain

ABSTRACT

In this work we propose a systematic procedure in order to integrate an arbitrary function over any chromaticity diagram similar to that associated with the CIE 1931 standard observer. The precision of this method is checked when smooth functions are considered.

1 Introduction

When analyzing in a rigorous way the statistics of color-matching experimental data, it has been pointed out the necessity for a feasible method in order to integrate over all the chromaticity diagram the probability density function obtained for the color matchings carried out in an experiment of chromatic discrimination [1, 2, 3, 4]. Of course, a such method can be useful in order to integrate any generic function defined over the chromaticity diagram.

As it is well known, different sets of color-matching functions generate different chromaticity diagrams. Perhaps, the major problem when we try to obtain a method of integration is its generalization in order to extend it to an arbitrary chromaticity diagram. In this work we propose a general method that can be directly applied to any chromaticity diagram similar to that generate by the CIE 1931 standard observer. The proposed procedure can be easily extended to any chromaticity diagram with an arbitrary shape which could be very different to that of the CIE 1931 chromaticity diagram.

Recibido: 15-7-99

2 Method of integration

Let $e_i(\lambda)$ be (i=1,2,3) a given set of color-matching functions associated with a fixed observer. The chromaticity diagram $D(\mathbf{x})$ generated by this set is defined by the surface delimited by the curve parameterized in λ

$$\mathbf{x}(\lambda) = (x_1(\lambda), x_2(\lambda)), \tag{1}$$

which contains the chromaticity coordinates

$$x_i(\lambda) = \frac{e_i(\lambda)}{\sum_{j=1}^{3} e_j(\lambda)}, \tag{2}$$

associated with the monochromatic stimuli as perceived by the considered observer. This surface is totally closed by the straight line d_3 joining the chromaticity coordinates,$(x_1^{(s)}, x_2^{(s)})$ and $(x_1^{(t)}, x_2^{(t)})$, of the monochromatic stimuli which define the lower and upper limits of the visible spectrum, with associated wavelengths λ_s and λ_t respectively. The chromaticity coordinates $\mathbf{x} = (x_1, x_2)$ of any color stimulus perceived by the observer are contained in this chromaticity diagram.

Let $g(\mathbf{x}) = g(x_1, x_2)$ be a continuous function in all the points of the chromaticity diagram, except in a set of zero measure. Our problem is to evaluate the integral

$$I = \int\int_{D(\mathbf{x})} g(x_1, x_2) dx_1 dx_2. \tag{3}$$

In general, we will have not an analytical expression for the color-matching functions. In this situation the major problem is to find the limits of the bi-dimensional integral (3). In order to solve this problem, the first step is to discretizate the chromaticity diagram in one dimension by splitting it in the x_1 axis in N intervals Δx_i in length $(i = 1, ..., N)$ as is shown in Fig. 1(a). In this case, expression (3) can be rewritten as

$$I = \lim_{\Delta x_i \to 0} \left(\sum_{i=1}^{N} \Delta x_i \int_{x_2^{(1)}(x_1^{(i)})}^{x_2^{(2)}(x_1^{(i)})} g(x_1^{(i)}, x_2) dx_2 \right), \tag{4}$$

where $x_1^{(i)}$ is the center of the $i-$th interval in which the x_1 axis has been splited, $x_2^{(1)}(x_1^{(i)})$ and $x_2^{(2)}(x_1^{(i)})$ being respectively the x_2 chromaticity coordinates for the lower and upper limits of the chromaticity diagram when $x_1 = x_1^{(i)}$. The followed procedure is illustrated in Fig. 1(a). If intervals Δx_i are small enough, expression (4) can be approximate with good precision by

$$I \simeq \sum_{i=1}^{N} \Delta x_i \int_{x_2^{(1)}(x_1^{(i)})}^{x_2^{(2)}(x_1^{(i)})} g(x_1^{(i)}, x_2) dx_2. \tag{5}$$

This expression allows us to reduce the two-dimensional integration process (3) to the calculus of N one-dimensional integrates which take the form

$$I_i = \int_{x_2^{(1)}(x_1^{(i)})}^{x_2^{(2)}(x_1^{(i)})} g(x_1^{(i)}, x_2) dx_2. \tag{6}$$

any case, it will not be a difficult task to extend the procedure to any chromaticity diagram generated by the set of color-matching functions of any real observer. The results obtained seem to indicate that, when considering smooth enough functions, the precision when calculating the integral is very good for a value $N = 2000$ of discrete intervals of integration. To use a larger value of N does not provide a noticeable improvement in the obtained precision. The method of integration has been used to compute color-difference thresholds [2, 3, 4] although other interesting application could be the computation of color-differences by using the interpolated expressions of the metric tensor g_{ij} given in Reference [12].

Acknowledgments

The authors want to express their gratitude to the anonymous referees for comments that helped us to improve the readability of this work.

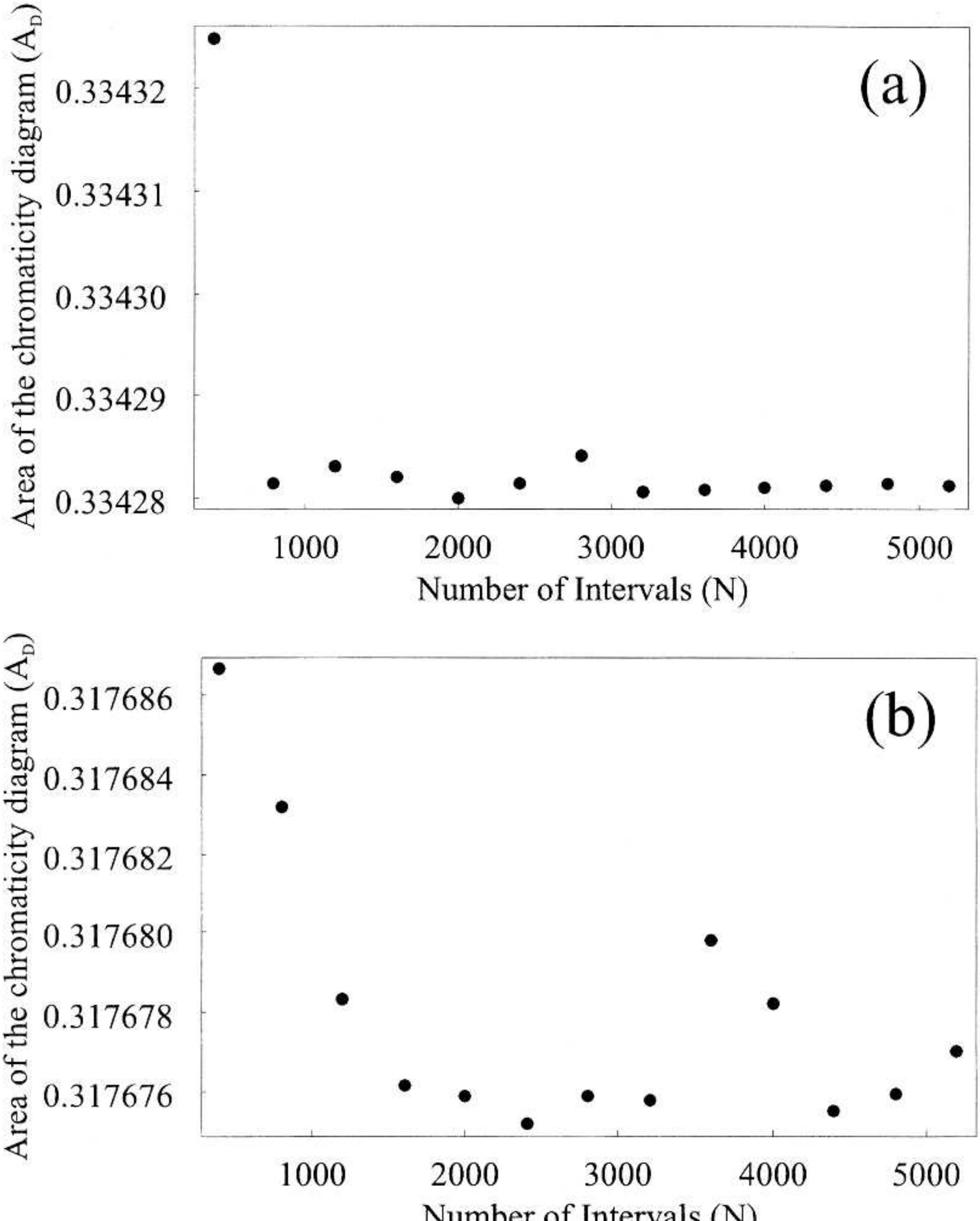

FIGURE 2. Areas of the CIE 1931 and CIE 1964 chromaticity diagrams ((a) and (b) respectively) as a function of the number N of intervals of integration in which the x_1 axis has been splited.

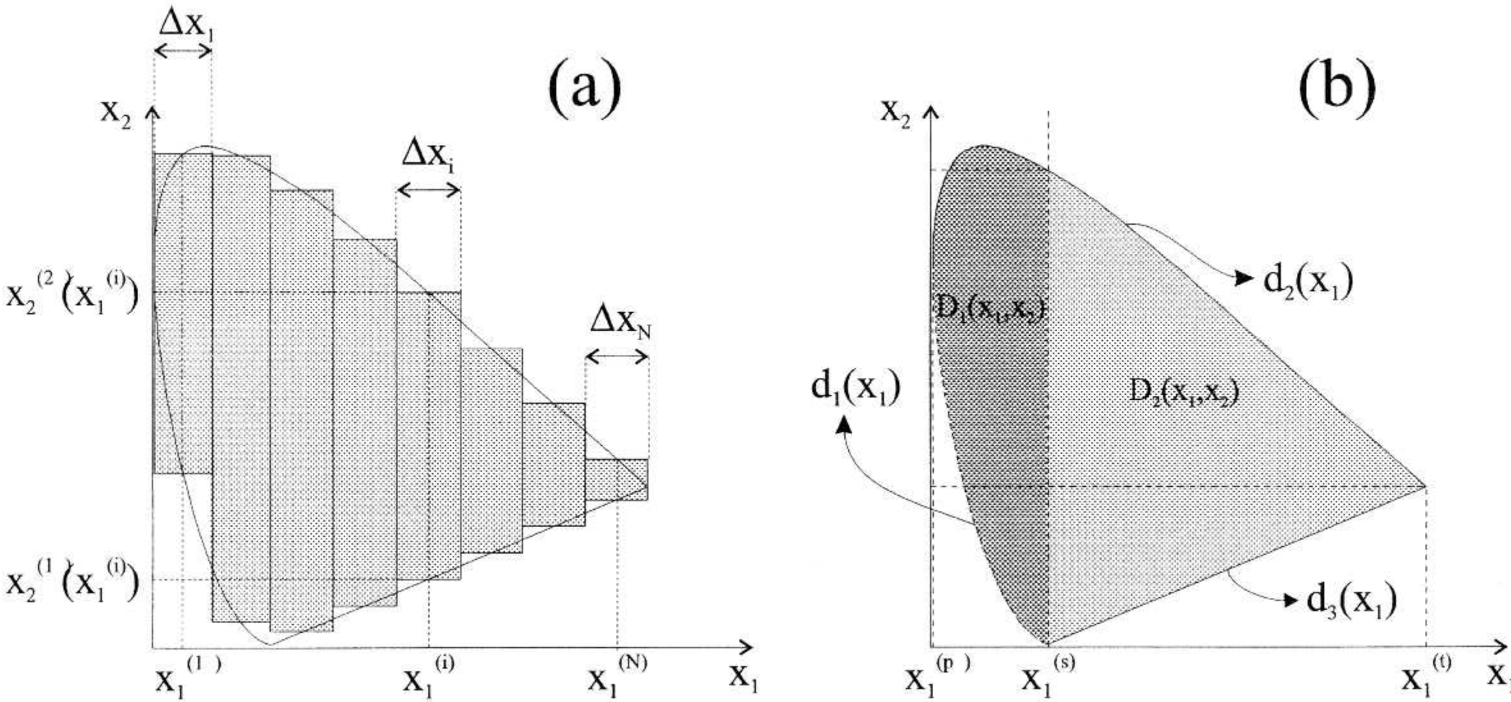

FIGURE 1. (a) One-dimensional discretization process over the chromaticity diagram, D, associated with the CIE 1931 standard observer. (b) Bijective parameterizations, d_1, d_2, and d_3, which define the limits of the considered chromaticity diagram. All the chromaticities can be splited in two regions D_1 and D_2 that $D = D_1 \cup D_2$.

- With results obtained in the last step, and using the three functions φ_i defined in expressions (26) to (28), we can compute the limits for the integration process in (32).

Obviously, the obtained precision when integrating (32) depends on the total number N of the intervals in which the x_1 axis has been splited. The larger N, the larger the precision. If we take $g(x_1, x_2) = 1$, integral (32) provides the area, A_D, of the chromaticity diagram generated by the considered set of color-matching functions, i.e.

$$A_D \simeq \sum_{i=1}^{N_1} \Delta x_i \left[\varphi_2(x_1^{(i)}) - \varphi_1(x_1^{(i)})\right] + \sum_{j=N_1+1}^{N} \Delta x_j \left[\varphi_2(x_1^{(j)}) - \varphi_1(x_1^{(j)})\right]. \tag{42}$$

In order to analyze the effect of the value of N in the precision when calculating integral (32), we have computed the area of the chromaticity diagram associated with the CIE 1931 standard observer for values of N in the interval $[400, 5200]$ by using equation (42). The results obtained are shown in Fig 2a: we can see that, when $N > 1800$, the maximum difference ΔA_D among the different values of the area A_D for different values of N is given by the quantity $\Delta A_D = 4 \times 10^{-6}$. This fact seems to indicate that the using values of N larger than 1800 does not provide a noticeable improvement in the precision when calculating the area of the chromaticity diagram. This analysis suggests us that, when working with smooth functions $g(x_1, x_2)$, to calculate integral (32) with values of N larger than 2000 does not provide a significative improvement in the obtained precision. A similar procedure has been carried out by using the color-matching functions of CIE 1964 standard observer and the results are given in Figure 2b.

Note that in the numerical examples presented in this work we have used the sets of color-matching functions provided in Reference [5]: these data are given with an accuracy of five digits and the sampling interval is $\Delta\lambda = 1$ nm. Although these sets are defined in the interval $\lambda \in [360, 830]$ nm, we have token $\lambda_s = 400$ and $\lambda_t = 700$ since this is the spectral interval in which the experimental color-matching functions are usually sampled (see for example References [6, 7, 8]).

It should be pointed out that the method used to reduce equation (4) to equation (5) can be improved by introducing weighting factors when computing the integral. We have chosen the described procedure to simplify the mathematical treatment. It becomes obvious that using other methods as those suggested in References [9, 10] will improve the accuracy. Nonetheless it has been shown in References [9, 10] how the accuracy is limited by the sampling interval to a great extent than the integration method selected. Equation (5) is a particular case of a Newton-Cotes method with weighting factors equal to unity: as it is indicated in [9, 10] if the function $g(x_1, x_2)$ is smooth enough the numerical error is negligible. On the other hand it is a common procedure in color science to replace summations by integrals (see for example References [6, 11]).

3 Conclusions

In this work we have developed a systematic procedure which allow us to carry out the integration process of an arbitrary function over all the chromaticity diagram. The method has been particularized to the case when we are considering diagrams similar to that generated by the set of color-matching functions associated with the CIE 1931 standard observer. In

In this case, the length of those interval contained in region $D_2(\mathbf{x})$ is

$$\Delta x_j = \frac{x_1^{(t)} - x_1^{(s)}}{N_2}. \tag{34}$$

If the lengths of the intervals in both regions are equals, condition

$$\frac{x_1^{(t)} - x_1^{(s)}}{x_1^{(s)} - x_1^{(p)}} = \frac{N_2}{N_1} \tag{35}$$

should be satisfied. Of course, the quotient in the left hand of this equation will not be, in general, an integer number. For this reason, we take

$$N_2 = I\left[\frac{x_1^{(t)} - x_1^{(s)}}{x_1^{(s)} - x_1^{(p)}}\right] N_1, \tag{36}$$

where operator $I[x]$ provides the integer part of x. In the case of the CIE 1931 standard observer, we have

$$\frac{x_1^{(t)} - x_1^{(s)}}{x_1^{(s)} - x_1^{(p)}} = 3.25, \tag{37}$$

and, by introducing this result in (36), we finally obtain

$$N_2 = 3N_1. \tag{38}$$

The relation between the lengths of the intervals in both regions is given by

$$\Delta x_j = 1.08\Delta x_i. \tag{39}$$

In the following we will describe, step by step, the process which systematize the procedure of the integration in (32):

- By using expression (36), for a fixed number N_1, we calculate the number N_2 of intervals in region $D_2(\mathbf{x})$. This step provides the total number of intervals in the x_1 axis.

- We obtain the lengths of the intervals in regions $D_1(\mathbf{x})$ and $D_2(\mathbf{x})$ from expressions (33) and (34) respectively.

- From expressions (7), (33), (34), and (36), the centers of the intervals of integration are calculated. These points will be given by

$$x_1^{(i)} = x_1^{(p)} + \frac{x_1^{(s)} - x_1^{(p)}}{N_1}\left(i - \frac{1}{2}\right), \text{with} i = 1, ..., N_1, \tag{40}$$

and

$$x_1^{(i+N_1)} = x_1^{(s)} + \frac{x_1^{(t)} - x_1^{(s)}}{N_2}\left(i - \frac{1}{2}\right), \text{with} i = 1, ..., N_2. \tag{41}$$

- By using expression (8), and taking into account that $\lambda \in [\lambda_s, \lambda_t]$, we compute the chromaticity coordinate $x_1^{(p)}$ which provides the minimum value of function $\tau_1^{(1)}(\lambda)$ in the interval $[\lambda_s, \lambda_t]$. In this process the wavelength λ_p is obtained too. From this second step we can calculate the point $\mathbf{x}^{(p)} = (x_1^{(p)}, x_2^{(p)})$, such that $x_i^{(p)} = \tau_i^{(1)}(\lambda_p)$.
- By introducing λ_s and λ_t in expressions (16), (18), and (19) points $\mathbf{x}^{(s)} = (x_1^{(s)}, x_2^{(s)})$ and $\mathbf{x}^{(t)} = (x_1^{(t)}, x_2^{(t)})$ are obtained. In this way, we have computed the intervals in which functions φ_i (i=1,2,3) are defined.
- From the results obtained in the previous step, we determine functions $\varphi_1(x_1)$ and $\varphi_2(x_1)$, given by expressions (26) and (27) respectively. These functions provide the parameterizations $\mathbf{d_1}(x_1)$ and $\mathbf{d_2}(x_1)$ defined in equations (23) and (24). Functions φ_1 and φ_2 are obtained by numerical interpolation of function $\varphi(x_1)$ defined in (20). The analytical expression for curve $\mathbf{d_3}(x_1)$ can be obtained from equations (25) and (28).

When the described procedure is applied to color-matching functions of CIE 1931 standard observer, the following results are obtained (although this set of function is defined in the interval $\lambda \in [360, 830]$ nm, we have token $\lambda_s = 400$ and $\lambda_t = 700$ nm):

$$\lambda_p = 504\,\text{nm}, \lambda_s = 360\,\text{nm}, \text{and} \lambda_t = 700\,\text{nm}, \tag{29}$$

$$\begin{aligned} (x_1^{(p)}, x_2^{(p)}) &= (0.00363, 0.63301), \\ (x_1^{(s)}, x_2^{(s)}) &= (0.17569, 0.79307), \\ (x_1^{(t)}, x_2^{(t)}) &= (0.73469, 0.26530). \end{aligned} \tag{30}$$

As it is shown in Fig. 1(b), the chromaticity diagram can be splited in two regions $D_1(\mathbf{x})$ and $D_2(\mathbf{x})$, such that condition

$$D(\mathbf{x}) = D_1(\mathbf{x}) \cup D_2(\mathbf{x}), \tag{31}$$

is satisfied. By taking into account definitions (23) to (25), it can be shown that region $D_1(\mathbf{x})$ is determined for those points $\mathbf{x}$ contained between curves $\mathbf{d_1}(x_1)$ and $\mathbf{d_2}(x_1)$ when $x_1 \leq x_1^{(s)}$. In a similar way, region $D_2(\mathbf{x})$ contains points contained between curves $\mathbf{d_3}(x_1)$ and $\mathbf{d_2}(x_1)$ with $x_1 > x_1^{(s)}$.

From decomposition (31), integral (5) can be rewritten as

$$I \simeq \sum_{i=1}^{N_1} \Delta x_i \int_{\varphi_1(x_1^{(i)})}^{\varphi_2(x_1^{(i)})} g(x_1^{(i)}, x_2) dx_2 + \sum_{j=N_1+1}^{N} \Delta x_j \int_{\varphi_3(x_1^{(j)})}^{\varphi_2(x_1^{(j)})} g(x_1^{(j)}, x_2) dx_2. \tag{32}$$

In this way, the integration process is splited in the calculus of N_1 integrates over the region $D_1(\mathbf{x})$ and another $N_2 = N - N_1$ integrates over the region $D_2(\mathbf{x})$. The computation of the limits of integration in equation (32) it is a trivial task by using parameterizations (23-25).

If we consider that all the intervals of integration contained in $D_1(\mathbf{x})$ have the same length and we fix the number N_1 of these intervals, the length will be given by

$$\Delta x_i = \frac{x_1^{(s)} - x_1^{(p)}}{N_1}. \tag{33}$$

in such a way that it satisfies

$$\varphi(\lambda) = \varphi[x_1(\lambda)] = \begin{cases} \varphi\left[\tau_1^{(1)}(\lambda)\right] = \tau_2^{(1)}(\lambda) \mathrm{forall} \lambda \in [\lambda_s, \lambda_t] \\ \varphi\left[\tau_1^{(2)}(\lambda)\right] = \tau_2^{(2)}(\lambda) \mathrm{forall} \lambda \in [0, 1]. \end{cases} \tag{21}$$

With this definition for function φ, the curve delimiting the chromaticity diagram is given by the following parameterization:

$$\mathbf{d}(x_1) = (x_1, \varphi(x_1)), \mathrm{with} x_1 \in [x_1^{(s)}, x_1^{(t)}]. \tag{22}$$

From a practical point of view, when computing the limits in integral (6) it will be useful to work with parametrizations for which condition $\varphi(x_1) \neq \varphi(x_2)$ implies that $x_1 \neq x_2$. Such a mapping will be referred to as a "valid parametrization". We can obtain the border of any chromaticity diagram similar to that associated with the CIE 1931 observer by using the valid parameterization shown in Fig. 1(b). These curves will be defined by the following expressions:

$$\mathbf{d_1}(x_1) = (x_1, \varphi_1(x_1)), \tag{23}$$

$$\mathbf{d_2}(x_1) = (x_1, \varphi_2(x_1)), \tag{24}$$

and

$$\mathbf{d_3}(x_1) = (x_1, \varphi_3(x_1)), \tag{25}$$

where

$$\varphi_1(x_1) = minc[\varphi(x_1)] \mathrm{with} x_1 \in [x_1^{(p)}, x_1^{(s)}], \tag{26}$$

$$\varphi_2(x_1) = maxc[\varphi(x_1)] \mathrm{with} x_1 \in [x_1^{(p)}, x_1^{(t)}], \tag{27}$$

and

$$\varphi_3(x_1) = \frac{x_2^{(t)} - x_2^{(s)}}{x_1^{(t)} - x_1^{(s)}} x_1 + \frac{x_1^{(t)} x_2^{(s)} - x_1^{(s)} x_2^{(t)}}{x_1^{(t)} - x_1^{(s)}} \mathrm{with} x_1 \in [x_1^{(s)}, x_1^{(t)}]. \tag{28}$$

Operators $minc[f(x)]$ and $maxc[f(x)]$ take respectively the minimum and maximum curve defined by the non-injective function $\varphi(x)$ in the prescribed interval. In these expressions the value of $x_1^{(p)}$ is given by relation (8).

Usually the sets of color-matching are tabulated in a discrete number of wavelengths. Functions φ_i defining curves $\mathbf{d_i}$ can be obtained by numerically interpolating color-matching functions $e_i(\lambda)$. Now we will describe the steps followed when obtaining the valid parameterizations which define the border of the chromaticity diagram:

- We have carried out a numerical interpolation of color-matching functions by using the popular Mathematica program in order to obtain functions $\tau_i^{(1)}(\lambda)$ as defined in (16). This program works by fitting polynomial curves of degree three between the specified points: in this way we obtain an approximate function with values $\tau_i^{(1)}$ at wavelengths $\lambda_s + i\Delta\lambda$, where i runs from 0 to 299 and $\Delta\lambda = \lambda_{i+1} - \lambda_i = 1$ nm. The precision of this method has been checked by interpolating a cosine function that has been previously sampled: the maximum difference between the interpolated function and the original function was 2.5×10^{-9}.

The lower limit in expression (6) is now obtained by finding the intersection of the straight line

$$\mathbf{x_i} = (x_1^{(i)}, x_2) \tag{14}$$

with the line $d_3(x_1)$ joining the chromaticities associated with the limits of the visible spectrum.

For a fixed size of Δx_i, the center of the intervals in which the chromaticity diagram has been splited are given by expression (7). Thus, the following step is to determine the limits in the integration process in (6). In order to do it, when $x_1^{(1)}$ is lesser than $x_1^{(s)}$, we need to solve in λ equations (11) and to substitute this result in equations (9) and (10). If $x_1^{(1)}$ is larger than $x_1^{(s)}$, it is necessary to find the value of λ which satisfies condition (11). In this case, by introducing this wavelength in expression (12) we can compute the upper limit in (6). The lower one is obtained by finding the intersection between the line (14) and $d_3(x_1)$.

If the set of color-matching functions is known, the line which determines the limits of the chromaticity diagram can be expressed by the curve parameterized in λ

$$\tau(\lambda) = \begin{cases} (\tau_1^{(1)}(\lambda), \tau_2^{(1)}(\lambda)) & if \quad \lambda \in [\lambda_s, \lambda_t] \\ (\tau_1^{(2)}(\lambda), \tau_2^{(2)}(\lambda)) & if \quad \lambda \in [0, 1], \end{cases} \tag{15}$$

with

$$\tau_i^{(1)}(\lambda) = \frac{e_i(\lambda)}{\sum_{j=1}^{3} e_j(\lambda)}, \tag{16}$$

and

$$\tau_i^{(2)}(\lambda) = x_i^{(s)} + (x_i^{(t)} - x_i^{(s)})\lambda, \tag{17}$$

with

$$x_i^{(s)} = \tau_i^{(1)}(\lambda_s), \tag{18}$$

$$x_i^{(t)} = \tau_i^{(1)}(\lambda_t), \tag{19}$$

λ_s and λ_t being the lower and upper limits of the visible spectrum. In equation (15) the parameter λ is only used to describe the chromaticity diagram in a strict mathematical sense. For this reason, we must not take into account its physical dimensions. In the last expression we have assumed that $\lambda \in [0, 1] \cup [\lambda_s, \lambda_t]$ is a parameter which is contained in a subset of the real numbers, but it has not dimensions of physical nature.

In order to integrate expression (6), it should be useful to look for a parameterization of the chromaticity diagram in the form $\mathbf{d} = (x_1, x_2(x_1))$ than to use parameterization (15). For a given set of color-matching functions, it is possible to find by using numerical techniques, the functional relation between the x_2 and x_1 coordinates for the monochromatic stimuli. Thus, we can build function

$$\varphi(x_1) = x_2(x_1), \tag{20}$$

Each one of the integrals I_i given by equation (6) are computed using a Newton-Cotes adaptative method with a tolerance of 10^{-8}. The N discrete intervals are centered at the points

$$x_1^{(i)} = x_1^{(p)} + \sum_{j=1}^{i-1} \Delta x_j + \frac{\Delta x_i}{2}, \tag{7}$$

where i runs from 1 to N, and

$$x_1^{(p)} = \min \left(\frac{e_1(\lambda)}{\sum_{j=1}^{3} e_j(\lambda)} \right), \tag{8}$$

as is shown in Fig. 1(b). If $x_1^{(i)}$ is lesser than $x_1^{(s)}$, the limits in equation (6) will be given by

$$x_2^{(1)}(x_1^{(i)}) = \min \left(\frac{e_2(\lambda_i^{(1)})}{\sum_{j=1}^{3} e_j(\lambda_i^{(1)})}, \frac{e_2(\lambda_i^{(2)})}{\sum_{j=1}^{3} e_j(\lambda_i^{(2)})} \right), \tag{9}$$

and

$$x_2^{(2)}(x_1^{(i)}) = \max \left(\frac{e_2(\lambda_i^{(1)})}{\sum_{j=1}^{3} e_j(\lambda_i^{(1)})}, \frac{e_2(\lambda_i^{(2)})}{\sum_{j=1}^{3} e_j(\lambda_i^{(2)})} \right), \tag{10}$$

$\lambda_i^{(1)}$ and $\lambda_i^{(2)}$ being the two wavelengths in the visible spectrum which satisfy

$$x_1^{(i)} = \frac{e_1(\lambda_i^{(1)})}{\sum_{j=1}^{3} e_j(\lambda_i^{(1)})} = \frac{e_1(\lambda_i^{(2)})}{\sum_{j=1}^{3} e_j(\lambda_i^{(2)})}. \tag{11}$$

In the case when $x_1^{(i)}$ is larger than $x_1^{(s)}$, the upper limit in the integration process described by (6) is given by

$$x_2^{(2)}(x_1^{(i)}) = \frac{e_2(\lambda_i^{(2)})}{\sum_{j=1}^{3} e_j(\lambda_i^{(2)})}, \tag{12}$$

where $\lambda_i^{(2)}$ is now the only wavelength in the visible spectrum satisfying condition

$$x_1^{(i)} = \frac{e_1(\lambda_i^{(2)})}{\sum_{j=1}^{3} e_j(\lambda_i^{(2)})}. \tag{13}$$

References

[1] J.M. Zoido, "Distancia estadística generalizada. La métrica del color", Ph. D. dissertation (Universidad Complutense de Madrid, Madrid, 1997).

[2] F. Carreño and J.M. Zoido, "Statistics of color-matching experimental data", Appl. Opt. **38**, 208-218 (1999).

[3] F. Carreño and J.M. Zoido, "The influence of luminance on color-difference thresholds" (1999). Submitted to Color Research.

[4] F. Carreño and J.M. Zoido, "Influencia de la luminancia en la asimetría de los umbrales de discriminación colorimétricos", V Congreso Nacional de Color, 213-214 (Tarrasa, 1999).

[5] G. Wyszecki and W.S. Stiles, *Color science: concepts and methods, quantitative data and formulae* (Wiley, New York, 1982)..

[6] F. J. M. Schmitt, "A method for the treatment of metamerism in colorimetry", J. Opt. Soc. Am. **66**, 601-608 (1976).

[7] W. A. Thorton, "Spectral sensitivities of the normal human visual system, color-matching functions and their principles, and how and why the two sets should coincide", Col. Res. Appl. **24** 139-156 (1999).

[8] F. Pérez-Ocón, E. Hita, L. Jiménez del Barco and J. L. Nieves, "Contribution to the experimental review of the colorimetric standard observer", Col. Res. Appl. **24** 377-388 (1999).

[9] B. Smith, C. Spiekermann and R. Sember, "Numerical methods for colorimetric calculations: a comparison of integration methods", Col. Res. Appl. **17** 384-393 (1992).

[10] B. Smith, C. Spiekermann and R. Sember, "Numerical methods for colorimetric calculations: sampling density requirements", Col. Res. Appl. **17** 394-401 (1992).

[11] P. D. Burns and R. S. Berns, "Error propagation analysis in color measurement and imaging", Col. Res. Appl. **22** 280-289 (1997).

[12] J. A. García, J. Romero, L. Jiménez del Barco and E. Hita, "Improved formula for evaluating color-differential thresholds", Appl. Opt. **31** 6292-6298 (1992).