ON MACADAM'S ELLIPSES

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RESUMEN

Se presenta un análisis sistemático de los resultados obtenidos por MacAdam [1]. Con la ayuda de un ordenador hemos repetido los cálculos usando las mismas ecuaciones y datos suministrados en su artículo. Entre los resultados obtenidos por nosotros y los proporcionados por MacAdam aparecen diferencias significativas. Estas discrepancias provienen del hecho de que las líneas rectas sobre las que se realizaron las diferentes igualaciones de color en torno a un estímulo cromático dado no tienen una intersección común que pueda ser considerado el centro de una elipse. Un estudio detallado de los datos nos lleva a cuestionar la geometría y el tamaño de las diferencias de cromaticidad igualmente perceptibles propuestas en el estudio original de MacAdam.

ABSTRACT

A systematic analysis of the results obtained by MacAdam [1] is presented. We repeat the calculations with the aid of a computer using the data supplied in his paper and the same equations. Significative differences emerge between the results obtained by us and those given by MacAdam. These discrepancies arise from the fact that the straight lines on which he carried out the different color matchings around a given chromaticity do not have a common intersection that could be considered as the center of the ellipse. A detailed study of the data leads us to question the geometry and the size of the curves of equally noticeable chromaticity differences proposed in MacAdam's original study.

1. INTRODUCTION

The quantitative evaluation of the visual system sensitivity to small variations of color stimuli has long been an important scientific problem. This interest arises from the need to

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solve technical problems derived from some industrial applications, mainly the establishment of adequate color difference formulas. The predictions made with color difference formulas must coincide, in so far as it is possible, with color discrimination experimental results obtained for normal observers. The first rigorous and systematic experimental investigation on visual sensitivity to small color differences was made by MacAdam [1].

As a result of that study, equal-noticeability color ellipses were established for 25 selected color stimuli (color centers). Each ellipse represents the region of equally noticeable chromaticity differences around its associated color center in the CIE 1931 chromaticity diagram. For a given color center, the noticeability of color differences in different directions was estimated from the standard deviation of the color matches made by the observer PGN. For a long time the results obtained by MacAdam have been considered the basis for the establishment and the evaluation of various color difference formulas proposed. The validity of these formulas has been conditioned to obtaining a good correlation between their predictions and the results obtained by MacAdam. These results were used to define empirical line elements that would lead to a description of color space geometry through the experimentally determined metric coefficients. Inductive line elements based on models of the visual system, have also been compared to these data. The results obtained by MacAdam in his rigorous and exhaustive study have been considered systematically as a standard reference for the comparison and the evaluation of later theoretical and empirical results in the fields of both the measurement and the specification of noticeable color differences detected by the visual system. There can be no doubt of the transcendence in the color science of the results and the conclusions provided by MacAdam's investigation.

A quantitative and systematic study of the visual system chromaticity discrimination requires a lot of color matchings in the vicinity of each selected color center. The realization of these measurements is a laborious task and it creates technical difficulties. Another difficulty arose from the later analysis of the great amount of data obtained during the observations, mainly considering that in those days neither personal computers nor hand calculators were available. All the calculations for the analysis of the experimental data were done by hand with the aid of a slide rule. The slide rule was specifically designed for the calculation of the synthesized chromaticities in the apparatus used by MacAdam. Some of the measurements were made on a large scale chromaticity diagram. These methods of analysis and treatment of experimental data can introduce errors which can accumulate in the calculation process and produce alterations in the results, mainly in their accuracy. In any event, such methods do not provide results as accurate as those obtained with modern computers.

In this paper we present a systematic analysis of the results obtained by MacAdam. We compare the results given in reference [1] with those we obtained using the original data provided by MacAdam in his paper. The objective of this study is to estimate the influence of the possible errors induced by the calculus methods and data analysis in the results given by MacAdam. We use the tristimulus values of the filters used (see Table 1 in reference [1]), the mean value θ of the observed angles of each measurement series, and the standard deviation $\Delta\theta$ (see Table III in reference [1]). From these data, and using the same equations given in MacAdam's paper, we repeated the calculations with the aid of a computer. The results obtained are compared to those provided in reference [1].

From the results obtained in the previous analysis it is obvious that from MacAdam's measurements it is not possible to deduce that the curves of equally noticeable chromaticity differences are ellipses. A question emerges that should be analyzed from both theoretical and experimental viewpoints:

• should the curves of equally noticeable chromaticity differences be greater than those given by MacAdam?

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The main discrepancy between the data provided by MacAdam and those obtained in this study is not due to the accumulation of errors in the calculation process used by MacAdam. The differences emerge from the hypothesis initially assumed by MacAdam related to the intersection in a common center of the different straight lines generated by the pairs of filters used in each color center. In the words of MacAdam «*The standard deviation of fifty observations was determined. This deviation was represented by two opposite radii from the center points towards the point representing the chromaticities of the two filters used for the synthesis. Each pair of filters used for the synthesis of the standard color resulted in the establishment of another diameter of the ellipse*».

In Section 12 we showed that the straight lines generated by each pair of filters around a color center (x_m, y_m) cannot be considered as common diameters of the same ellipse. This consideration should be valid only if the ellipses obtained by MacAdam were of such a size that the intersections shown in Figs. 10 and 11 could be considered as a unique point. The distances from the center (x_m, y_m) to the intersections of the straight lines are greater than the size of the corresponding ellipse given by MacAdam. The spatial dispersion of the intersections around the color center considered as a common point by MacAdam is greater than the distance Δs_m he obtained as a result of his measurements.

From the aforementioned analysis we conclude that the problem emerges from the unchecked initial assumptions related to the intersections of the straight lines, although the measurements were carefully made. The filters, although carefully prepared and characterized, were not selected accurately enough to derive the standard deviations around a unique color center from the measurements carried out with them.

From the distribution of points, $(x_e, y_e)^s$ and $(x_e, y_e)^l$, it cannot be concluded that the ellipses provided by MacAdam correspond to curves of equally noticeable chromaticity differences. Curves that should contain inside the distributions of points obtained by us, should be substantially greater than those given by MacAdam. Figure 10 clearly shows this point: MacAdam's ellipses are drawn with the semiaxes multiplied by 10 together with the points $(x_e, y_e)^s$ and $(x_e, y_e)^l$. It seems that the curves of equally noticeable chromaticity differences that could be derived from the measurements made by MacAdam should be of a size greater than that provided in reference [1].

The results of the previous Sections imply other conclusions obtained afterwards by MacAdam and other authors. Since it cannot be concluded from MacAdam's experiment that his ellipses represent curves of equally noticeable chromaticity differences, then the metric coefficients g_{ij} of the empiric line element proposed by MacAdam in a later study [12] are not adequate to describe the geometry of the space of color representation. Assuming that the distribution of the standard deviations obtained in MacAdam's experiment is normally distributed, Silberstein and MacAdam [13] theoretically deduce that the curves of equally noticeable chromaticity differences must be ellipses. They computed the parameters of the equal-noticeability ellipses from the experimental data obtained by MacAdam. The similarity of the results obtained with the ellipses given by MacAdam lead them to conclude that the initial normality hypothesis of the data is correct. Since the observations made by MacAdam around a color center do not have a common center, it does not seem well justified to speak of distributions around a point. The normality hypothesis, supposedly «confirmed», was assumed in a later study by Brown and MacAdam [14].

We have observed that in most cases the quotients e_{l1} and e_{l2} are significantly greater than unity, thus indicating that the intersections are outside and at a great distance from the edge of MacAdam's corresponding ellipse. It should be pointed out that some intersections are off the chromaticity diagram, as is the case of intersection number 36 in ellipse number 6 ($x_l = 0.712$ and $y_l = 2.280$), and intersection number 17 in ellipse number 25 ($x_l = 0.597$ and $y_l = -0.422$). The mean values of quotients e_{l1} and e_{l2} for ellipse 4 are $\hat{e} = 4.4$ and $\hat{e} = 6.4$ respectively. The straight lines joining each pair of filters and their intersections, the points that we have calculated ($(x_e, y_e)^s$ and $(x_e, y_e)^l$) and MacAdam's corresponding ellipses are drawn in Fig. 11 for all the color centers.

Obviously the intersections cannot be considered to have a common center in practice. MacAdam assumed the situation represented in Fig. 3a but in practice this situation does not arise. This is the cause of the differences we found in Section 8 between the chromaticity coordinates (x,y) and the center (x_m, y_m) given by MacAdam.

13. DISCUSSION

The results obtained in Sections 6 and 7 clearly show that the condition of constant luminance is guaranteed in MacAdam's experiment within a margin of error of 5%.

The standard deviations of chromaticity were accurately computed in all the series of color matchings (see Section 10). The standard deviations of the chromaticity coordinates, Δx and Δy, were also well computed (see Section 11). These two facts lead us to conclude that, with the definition criterion of chromaticity noticeability established by MacAdam, the standard deviations Δs_m provide an accurate *local* measurement of the sensitivity to small color differences around a color center *(x,y) in the direction* in which they were determined.

From the results obtained in Section 8 we conclude that most of the computed color matchings (x,y) are outside the corresponding ellipse given by MacAdam (see Figs. 9 and 11). Thus from the measurements carried out by MacAdam, it is not possible to infer the existence of a common center to represent the standard deviations of chromaticity in all the directions in the chromaticity diagram. For this reason the computed distribution of points of equally noticeable chromaticity differences around a fixed point, $(x_e, y_e)^s$ and $(x_e, y_e)^l$, is far from being on the border of an ellipse (see Figs. 9 and 11). The dispersion of the distribution of these points around (x_m, y_m) is greater and less systematic than that proposed by MacAdam.

MacAdam states that each pair of filters is capable of reproducing a color stimulus very close to the chromaticity given by (x_m, y_m): «*The ellipses shown in Figs. 23 to 47 represent the noticeability of chromaticity variations in all directions from the chromaticities indicated at the centers of the ellipses. The observations represented by each ellipse were obtained by selecting from five to eight pairs of filters, each pair of which could be used in the instrument to synthesize a chromaticity very closely approximating that represented by the center of the ellipse*». The proximity assumed by MacAdam is not enough to determine the standard deviations of chromaticity Δs_m. The magnitude that he tries to deduce from his measurements is, after the calculations, smaller than the spatial uncertainty in the determination of the color center (see Table 2 and Fig. 7).

all the straight lines joining the filters used in a certain color center is a necessary condition for a common center (see Fig. 3a). The intersection of pairs of straight lines, within a certain margin of error, should be sufficiently close to each other in order to consider that there exists a unique point (see Fig. 6).

The equation of the straight line generated by each pair of filters is determined by the chromaticity coordinates of the filters. The point *(x,y)* is contained in the corresponding straight line generated by the filters. There are some color centers where the color matchings are repeated in certain directions. In ellipse number 4 the series of equalizations 1 and 2 are made with the same pair of filters (86 and 14), thus the straight line generated by the filters is the same in both cases. This also occurs for series 5 and 6 (filters 71 and 24), 7 and 8 (filters 78 and 14), 9 and 10 (filters 44 and 45) and 11 and 12 (filters 72 and 15) (see Table III in page 260, reference [1]). In these cases there are different points *(x,y)* on the same straight line corresponding to the different average equalizations obtained. This situation obviously reduces the number of straight lines generated with regard to the number of series of equalizations. When the previously mentioned situation is produced we used only the first series of equalizations, in the order provided by MacAdam, eliminating the rest of the repeated series of data. For each ellipse we have obtained the equation of all the straight lines generated by the pairs of filters and we have computed the intersections between each pair (x_I, y_I). In each intersection, we will denote as the first straight line the line generated by the series of measurements that appears first according to the order provided by MacAdam. We could consider that the intersections have a common point, (x_m, y_m), if the distance from the intersections to this point is small enough, following a similar criterion as that in Section 8. We have computed the distance from the color center provided by MacAdam to each intersection, d_I, as

$$d_I = \sqrt{(x_m - x_I)^2 + (y_m - y_I)^2} \tag{24}$$

If all the intersections are, at least, inside MacAdam's corresponding ellipse, then a common point could be assigned to them. We have computed the quotients of the distances d_I with regard to the standard deviations of chromaticity Δs_m as

$$e_{I1} = \frac{d_I}{(\Delta s_m)_1} \tag{25}$$

and

$$e_{I2} = \frac{d_I}{(\Delta s_m)_2} \tag{26}$$

$(\Delta s_m)_1$ and $(\Delta s_m)_2$ being the standard deviations of chromaticity given by MacAdam on the first and second straight lines that produced the intersection respectively. The quotients given by equations (25) and (26) should be less than unity in order to consider that the intersection is inside the corresponding ellipse. We have computed the distances d_I and the quotients e_{I1} and e_{I2} for the intersections of the straight lines of all the ellipses. The results obtained for ellipse 4 are listed in Table III.

reference [22]. It should be pointed out that around the selected chromaticity, the points representing equal noticeability of color differences vary greatly from those provided by MacAdam (see Fig. 4). This behaviour is systematically reproduced in the rest of the ellipses. The distributions of points $(x_e, y_e)^s$ and $(x_e, y_e)^l$ are shown for all the ellipses in Figs. 10 and 11 (solid points) compared with MacAdam's corresponding ellipses.

The discrepancies that emerge in the data given by MacAdam and those obtained by us could be attributed to the results obtained in Section 8. The influence of the differences between the values of Δs_m supplied by MacAdam and the calculated standard deviations Δs should also be considered (see Section 10). To analyze this second possibility we represent the standard deviations of chromaticity (Δs) as two points $(x_h, y_h)^s$ and $(x_h, y_h)^l$ in the chromaticity diagram. These points are obtained considering that Δs can be computed as distances to a common center for all the directions in a given ellipse. We choose the center as the point given by MacAdam (x_m, y_m). In other words, the results obtained in Section 8 will be ignored in the following analysis. We will take into account only the differences that we have found between Δs_m and Δs and the differences between m and f_m. In this case, points $(x_h, y_h)^s$ and $(x_h, y_h)^l$ are given by

$$(x_h, y_h)^S = (x_m + \frac{m}{|m|} |\Delta x|, y_m + |\Delta y|)$$

and (23)

$$(x_h, y_h)^l = (x_m - \frac{m}{|m|} |\Delta x|, y_m - |\Delta y|)$$

The standard deviations of the chromaticity coordinates are computed from equations (19) and (21), m being the slope computed in Section 9. It should be pointed out that the distribution of the points obtained in this way for all the color centers is similar to the distribution of points $(x_e, y_e)^s_m$ and $(x_e, y_e)^l_m$ given by MacAdam. If we only consider the differences in the standard deviations of chromaticity and those existing in the slopes, the discrepancies with the results given by MacAdam are of secondary importance. This point confirms that the treatment of the experimental data made by us and that of MacAdam lead to the same results when we consider both the slope of the straight lines and the standard deviations of chromaticity.

The last analysis makes it clear that the differences between the distributions of points given by MacAdam ($(x_e, y_e)^s_m$ and $(x_e, y_e)^l_m$) and those obtained by us ($(x_e, y_e)^s$ and $(x_e, y_e)^l$) arise from the results obtained in Section 8. The real dispersion of points of equal noticeability of color differences around the chromaticity specified by point (x_m, y_m) is greater and less systematic than that given by MacAdam and, what is more, it does not seem that the distribution of points $(x_e, y_e)^s$ and $(x_e, y_e)^l$ around the color center (x_m, y_m) fits the equation of an ellipse.

12. ANALYSIS OF THE INTERSECTIONS AMONG THE STRAIGHT LINES OF THE FILTERS

In this Section we analyze the distance of the intersection point of each pair of straight lines to the color center (x_m, y_m) for all the ellipses. The intersection at a common point of

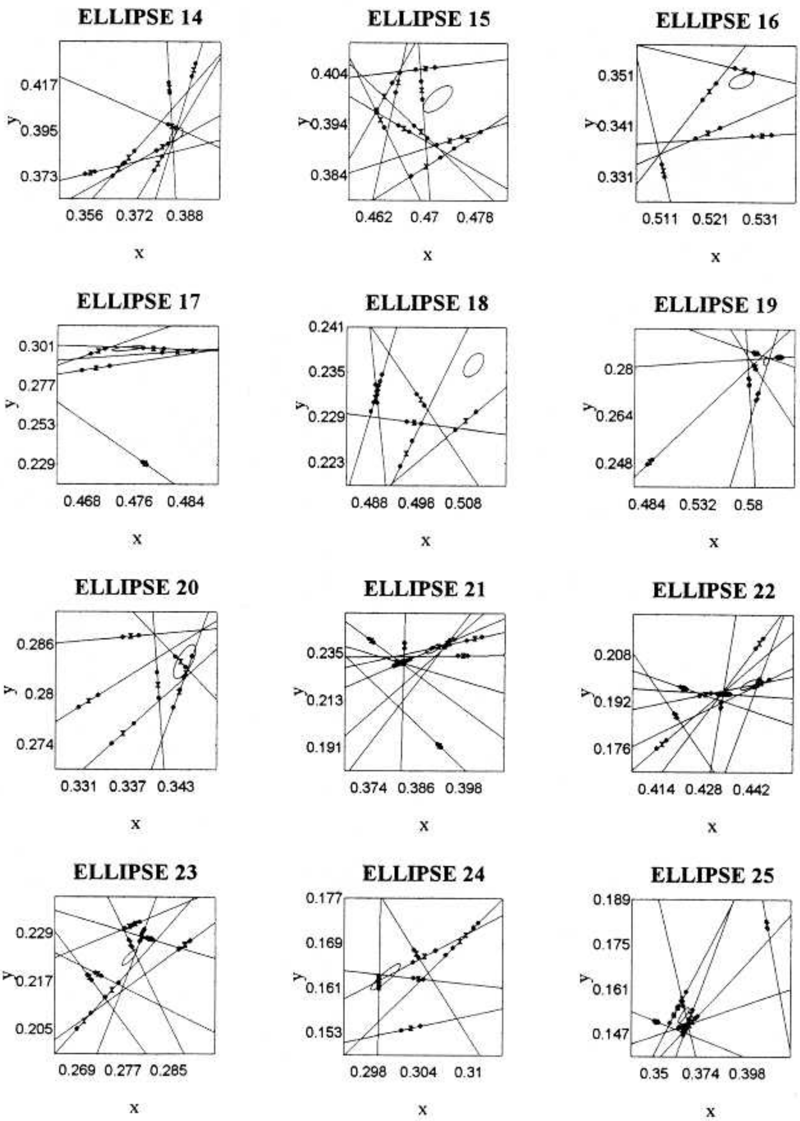

FIGURE 11(b).

FIGURE 11. Distribution of points of equally noticeable chromaticity differences $(x_e, y_e)^S$ and $(x_e, y_e)^I$ computed for all the color centers (solid points) compared with MacAdam's corresponding ellipse except for ellipse 4. The straight lines generated by each pair of filters are also drawn. The points labeled with an **X** correspond to the average chromaticity computed for each series.

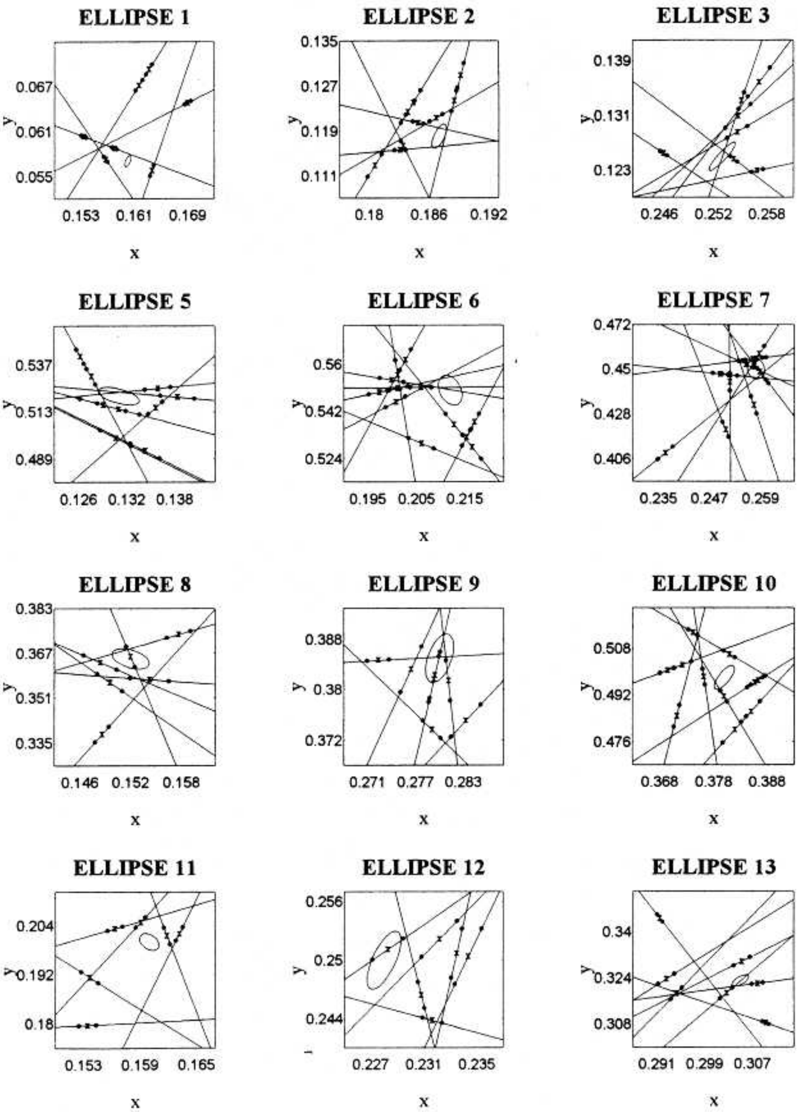

FIGURE 11(a).

FIGURE 11. Distribution of points of equally noticeable chromaticity differences $(x_e, y_e)^S$ and $(x_e, y_e)^I$ computed for all the color centers (solid points) compared with MacAdam's corresponding ellipse except for ellipse 4. The straight lines generated by each pair of filters are also drawn. The points labeled with an **X** correspond to the average chromaticity computed for each series.

The calculations are carried out for all the directions in each ellipse. The results obtained for ellipse number 4 are shown in Fig. 9 (solid points), together with MacAdam's corresponding ellipse. Figure 10 shows points $(x_e, y_e)^s$ and $(x_e, y_e)^l$ for all the ellipses. In this figure the ellipses are drawn with the semiaxes 10 times greater than those provided in

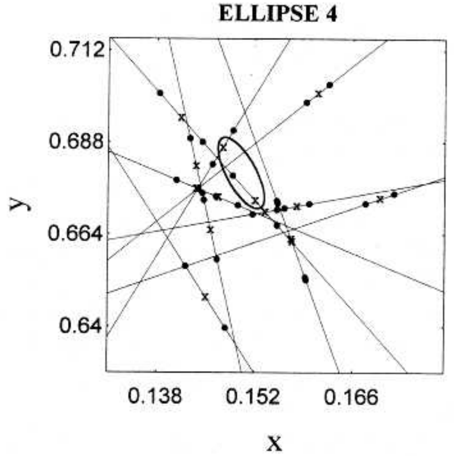

FIGURE 9. Distribution of points of equally noticeable chromaticity differences $(x_e, y_e)^S$ and $(x_e, y_e)^l$ computed for color center 4 (solid points) compared with MacAdam's corresponding ellipse. The straight lines generated by each pair of filters are also drawn. The point labeled with an X corresponds to the average chromaticity (x, y) computed for each series.

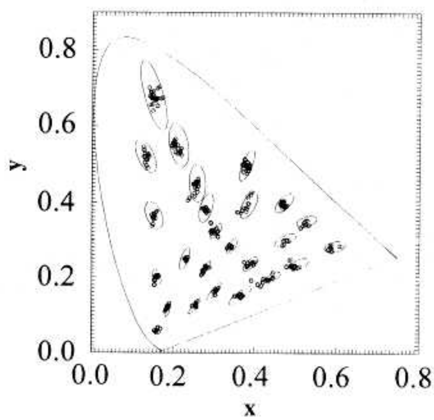

FIGURE 10. Representation in the chromaticity diagram of the distributions of points of equally noticeable chromaticity differences $(x_e, y_e)^S$ and $(x_e, y_e)^l$ around each color center. These distributions are compared with MacAdam's corresponding ellipses. The ellipses have been drawn with the semiaxes ten times greater than those provided in reference [22].

data with regard to our results as $\varepsilon\,(\Delta s_m) = \frac{\Delta s_m - \Delta s}{\Delta s}\,100$. Figure 8 shows the results for the ellipse number 4. We have observed that in most of the cases there is no appreciable difference between the data that we have calculated and those provided by MacAdam.

11. ANALYSIS OF THE CHROMATICITY COORDINATES OF THE POINTS THAT CONFIGURE THE ELLIPSES

Using the coordinates (x, y) (see Section 8), the slopes m of the straight lines that join the filters (see Section 9) and the standard deviations Δs (see Section 10), we obtain the points in the chromaticity diagram corresponding to equal noticeability of color differences in a given direction around the color center of interest. These points, $(x_e, y_e)^s$ and $(x_e, y_e)^l$, are conceptually equivalent to the points $(x_e, y_e)^s_m$ and $(x_e, y_e)^l_m$ that configure MacAdam's ellipses (see Section 5). The only difference among them arises from the fact that they have been calculated following different methods in the treatment of the experimental data. The purpose of this section is to evaluate the magnitude and importance of the possible differences that can emerge among them.

We showed in Section 8 that the points with coordinates (x,y) did not have a common center. Under these circumstances, the standard deviations of chromaticity, Δs, obtained in different directions cannot be computed as distances on each straight line with regard to a common center in a given ellipse. These distances must be computed on the straight line generated by each pair of filters, from point (x,y) which is determined by series of equalizations carried out in that direction (see Fig. 6). In the following analysis, we assume as did MacAdam, the hypothesis that the noticeability of color differences around a color center is, in each direction, independent of the way in which the chromaticity variation is produced. In this case, in each series of equalizations carried out with a pair of filters, the computed standard deviation, Δs, is represented by two points $(x_e, y_e)^s$ and $(x_e, y_e)^l$, situated on the straight line that joins the filters. Both points are situated in opposite directions at a distance Δs from the point (x,y). The relation between the standard deviations of the chromaticity coordinates, Δx and Δy, is given by

$$\Delta y = m\Delta x \tag{19}$$

The standard deviation of chromaticity will be given by

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2, \tag{20}$$

and

$$\left|\Delta x\right| = \sqrt{\frac{(\Delta s)^2}{1+m^2}}\,. \tag{21}$$

Points $(x_e, y_e)^s$ and $(x_e, y_e)^l$ are calculated according to the sign of the slope m,

$$(x_e, y_e)^S = (x + \tfrac{m}{|m|}|\Delta x|,\ y + |\Delta y|)$$

and

$$(x_e, y_e)^I = (x - \tfrac{m}{|m|}|\Delta x|,\ y - |\Delta y|) \tag{22}$$

$$m = \frac{y_v - y_u}{x_v - x_u}, \quad (17)$$

(x_u, y_u) and (x_v, y_v) being the chromaticity coordinates of the filters. From the tristimulus values given by MacAdam (see columns 4, 5 and 6 of Table I in reference [1]) we determine the «mass» given to each filter, m_u and m_v, and its chromaticity coordinates. Taking into account the filters used, we compute the slope of the straight lines associated with the color matchings. We have computed the quotient of the slope given by MacAdam, f_m, and the slope given by equation (17). We observe that for all the ellipses the quotients m/f_m are very close to unity. The mean value of these quotients for ellipse 4 is $\widehat{m/f_m} = 1.0$ with standard deviation $\delta(\widehat{m/f_m}) = 0.2$. The slopes computed using equation (17) are practically coincidental with those slopes given by MacAdam.

10. ANALYSIS OF THE STANDARD DEVIATION Δs_m

The chromaticity deviations (Δs_m) in each series of observations around a color center and for a given direction are calculated by MacAdam with the aid of a slide rule and measurements made on a large scale chromaticity diagram [1]. In this section, we compute these magnitudes from the same equations given in reference [1] using a computer. We compute S from equations (2) and (4). The associated standard deviation, Δ s, is given by equation (6), $\Delta\theta$ being the standard deviation of the mean angle obtained in each series of measurements (see column 4 in Table III of reference [1]) and

$$\frac{df(\theta)}{d\theta} = \frac{2r \sin\theta \cos\theta}{(\sin^2\theta + r\cos^2\theta)^2} \quad (18)$$

For each ellipse we computed the standard deviation Δs for the observations carried out in the different directions. We have computed the percentage of deviation of MacAdam's

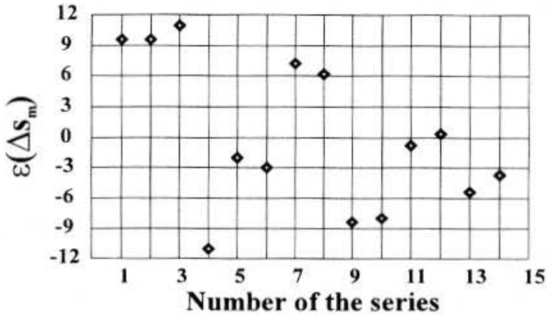

FIGURE 8. Percentage of deviations $\varepsilon(\Delta s_m)$ of the standard deviation of chromaticity given by MacAdam with regard to that computed by us for the different series of observations carried out in ellipse 4. The X axis of the figure indicates the number of the series of observations.

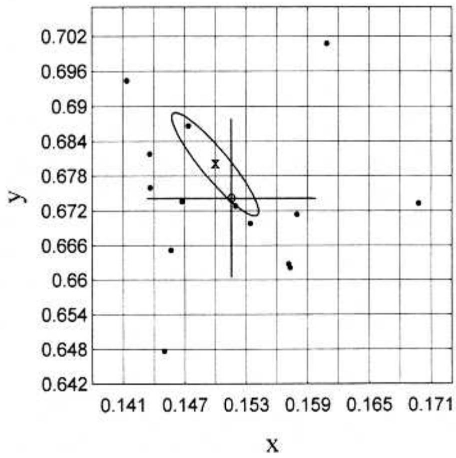

FIGURE 7. Chromaticity coordinates *(x, y)* of the average chromaticities obtained in each series of color matchings around color center 4 (solid points). These points are compared with MacAdam's corresponding ellipse. The center of this ellipse given by MacAdam (x_m, y_m) is labeled with an **X**. The computed average center (x_c, y_c) (unfilled circle) and its associated standard deviations δx_c and δy_c in each direction are also represented (solid lines parallel to the axis of the figure).

is bound by unity. We have computed magnitudes d and e in each direction for all the ellipses. The results for ellipse 4 are listed in Table II. In most of the observations carried out by MacAdam, the distance from (x,y) to (x_m, y_m) is greater than the standard deviation of the equalization Δs_m given by MacAdam in the corresponding direction. In most cases the value of e is considerably greater than unity. This situation is illustrated in Fig. 7 for ellipse number 4, where the chromaticity coordinates (x,y), the center given by MacAdam (x_m, y_m) and the «average center» (x_c, y_c) are drawn. In this figure and using the data provided in reference [22] we have also plotted the ellipse obtained by MacAdam for this color center. We have computed the mean value and the standard deviation of the quotients e for each ellipse. The results obtained for ellipse 4 are $\hat{e} = 3$ and $\delta\hat{e} = 2$. In all cases, the mean values of $\hat{e}$ are greater than unity. The maximum mean value is obtained for ellipse 17 ($\hat{e} = 16$ and $\delta\hat{e} = 29$).

It is clear from the previous results that the chromaticity coordinates (x,y) obtained in each direction for a given ellipse cannot be considered as a unique point, i.e., the point (x_m, y_m) given by MacAdam cannot be considered as a common point for all the equalizations carried out in the different directions.

9. ANALYSIS OF THE SLOPE OF THE STRAIGHT LINES GENERATED BY THE FILTERS

The chromaticity coordinates of the color matchings made by the observer in each series of measurements lie on the straight line that joins the chromaticities of the filters used. The slope of this line is given by achieved on replacing the «average center» by the center given by MacAdam (x_m, y_m) if points (x,y) could be considered as a unique point.

The differences between the coordinates (x,y) and MacAdam's centers (x_m, y_m) could arise from the fact that, in practice, the lines that join the different pairs of filters used around each color center do not have a common intersection (see Fig. 6). We will analyze this point later in this study (see Section 12). The differences will be of no importance if the distances from these points (x,y) to the center of the ellipse are small enough. In this case the points (x,y) can be considered as a unique point (x_m, y_m).

For each ellipse we have computed distance d between the center given by MacAdam (x_m, y_m) and each one of the points (x,y) as follows

$$d = \sqrt{(x_m - x)^2 + (y_m - y)^2} \tag{15}$$

Distance d given in equation (15) is computed from the data obtained by us (see Table I for ellipse 4) and the centers given by MacAdam (see columns 5 and 6 in Table III, reference [1]). For a given ellipse, the points (x,y) should be considered as a unique point if and only if the distance d is much less than the size of the ellipse. In other words: these points should be contained inside the ellipse. This condition is satisfied if the following quotient

$$e = \frac{d}{\Delta s_m} \tag{16}$$

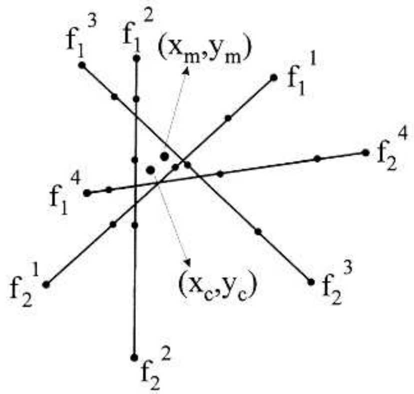

FIGURE 6. Spatial location of the straight lines generated by each different pair of filters around a color center. The average center (x_c, y_c) is not coincidental with the center (x_m, y_m) given by MacAdam. The notation for the filters is the same as in Fig. 3(a).

tristimulus values of the filters used we computed the chromaticity coordinates associated with each mean angle. Since for each ellipse MacAdam computes all the points $(x_e, y_e)^s{}_m$ and $(x_e, y_e)^I{}_m$ from a common center (see Section 5), the chromaticity coordinates computed by us (x,y) should coincide, or be very close to the corresponding point (x_m, y_m) considered as the center of the ellipse by MacAdam.

MacAdam assumed that the chromaticity coordinates associated with the mean angle obtained are the same in all the directions of the ellipse and that they are coincidental with the center of the ellipse itself (see columns 5 and 6 in Table III of reference [1]). The chromaticity coordinates obtained by us, (x,y), for the mean angle in each direction for a fixed ellipse are different among themselves and, generally, do not coincide with the coordinates of the center provided by MacAdam (x_m, y_m). The results obtained for ellipse 4 are shown in Table I. We compute the percentage of deviation of the chromaticity coordinates given by MacAdam with regard to those we have computed as follows

$$\varepsilon(x_m) = \frac{x - x_m}{x} 100, \qquad \varepsilon(y_m) = \frac{y - y_m}{y} 100. \tag{13}$$

Although the aforementioned percentage of deviation is not very large, sometimes the deviations are greater than 10%. Obviously the chromaticity coordinates of the different series of equalizations show discrepancies among themselves. If the distance in the chromaticity diagram among the points (x,y) obtained in the different directions is not very big, they could be considered as a unique point (the center of the ellipse). It is a reasonable criterion to consider this point as the one with chromaticity coordinates $\left(x_c = \frac{\sum_{j=1}^{n} x_j}{n}, y_c = \frac{\sum_{j=1}^{n} y_j}{n} \right)$, x_j and y_j being the coordinates associated with the *j-th* series of observations carried out in the ellipse and n being the total number of series of observations made in each ellipse. We have computed the chromaticity coordinates of the mean center for all the ellipses and the standard deviations δx_c and δy_c associated to x_c and y_c respectively. The results obtained for ellipse 4 are $x_c = 0.152$, $\delta x_c = 0.008$ (5.4%), $y_c = 0.674$ $\delta y_c = 0.014$ (2%), whereas the chromaticity coordinates given by MacAdam for this color center are $x_m=0.150$ and $y_m=0.680$.

Most of the calculated «average centers», (x_c, y_c) do not coincide with the centers given by MacAdam (x_m, y_m). Nevertheless the following inequalities hold for all the ellipses, except for ellipses 9, 12 and 18,

$$x_c - \delta x_c \leq x_m \leq x_c + \delta x_c$$

and

$$y_c - \delta y_c \leq y_m \leq y_c + \delta y_c \tag{14}$$

The coordinates of the color centers given by MacAdam, coincide within the estimated margin of error with the computed «average centers» (x_c, y_c). A reduced error will be

From the previous analysis it may be concluded that the condition of constant luminance, with regard to the filters, is guaranteed in most of the observations within a margin of error of about 5%.

7. ANALYSIS OF THE LUMINANCE IN THE MATCHINGS CARRIED OUT IN EACH ELLIPSE

In this section we analyze the condition of constant luminance in the equalizations made by PGN in the different directions for each of the ellipses. Using equation (2) we have computed the Y tristimulus value associated with the mean value of the angle, θ (see 3rd column of Table III in reference [1]). For each ellipse, the mean value of this magnitude ($\hat{Y}$) provides us with an estimation of the mean luminance of the equalizations made around that color center. The deviation ΔY of the tristimulus values obtained in the different series of measurements with regard to the mean value, indicates the uniformity of luminance in the equalizations made for each ellipse. We will compute this deviation as $\Delta Y = \frac{\hat{Y}-Y_f}{\hat{Y}} 100$. In none of the 25 color centers do these deviations go as high as 10%, in fact they are below 5%. We have computed the mean value, $\hat{Y}$, and its standard deviation, $\delta\hat{Y}$, for each color center. The results obtained for ellipse 4 are $\hat{Y} = 0.00282$ and $\delta\hat{Y} = 0.00008$ (3%). Taking the standard deviation as the error associated with $\hat{Y}$, it should be pointed out that the mean luminance in the observations is within an interval of 5%, except for ellipses number 11 and 12 ($\delta Y = 6\%$). Considering the associated error interval, the mean luminances estimated, except for those of ellipses 2, 3 and 6, coincide with the mean luminous transmittance estimated for the filters, $\hat{Y}_f$,i.e.,

$$\hat{Y} - \delta Y \leq \hat{Y}_f \leq \hat{Y} + \delta Y \tag{11}$$

Taking $\delta Y_f = \pm\ 0.00015$ (see Section 6), the mean luminance for each ellipse coincides within this margin of error with $\hat{Y}_f$, i.e., for each color center the following condition is satisfied

$$\hat{Y}_f - \delta\hat{Y}_f \leq \hat{Y} \leq \hat{Y}_f + \delta\hat{Y}_f \tag{12}$$

These results again confirm the accuracy in the condition of constant luminance achieved in MacAdam's experiment.

8. ANALYSIS OF THE MEAN CHROMATICITIES FOR EACH COLOR MATCHING

We compute the tristimulus values *X, Y* and *Z* associated with the mean angle obtained in each series of observations using equation (2). We use the value of θ given by MacAdam (column 3 of Table III in reference [1]) and the tristimulus values of the pair of filters used in each series of equalizations (columns 4, 5 and 6 of Table I in reference [1]). For each ellipse, the pair of filters used in the observations carried out in the different directions is specified in columns 1 and 2 of Table III in reference [1]. Taking into account the

The superindex S is used for the points whose chromaticity coordinate y increases and the superindex I is used for those whose chromaticity coordinate y decreases. The points obtained in this way are similar to those graphically provided by MacAdam (Figs. 23 to 47 in reference [1]). The result obtained for ellipse 4, with the straight lines calculated from the slope provided by MacAdam, are shown in Fig. 4.

The numeration followed for the ellipses in this study is coincidental with the order given in reference [22]. To produce all the figures which show MacAdam's ellipses, we have used the data provided in reference [22].

6. ANALYSIS OF THE LUMINOUS TRANSMITTANCE OF THE FILTERS

MacAdam provides the tristimulus values (X_f, Y_f, Z_f) for the 105 filters used in his investigation (see Table I in reference [1]). The condition of constant luminance in all the observations carried out was conditioned to the uniformity in the value of the luminous transmittance Y_f, of the filters used. We have calculated the mean value of the Y tristimulus value for all the filters. Taking the standard deviation (δY_f) as the error associated with the mean value of a magnitude, we conclude that the mean luminous transmittance of the filters is $\hat{Y}$ = (0.00285 ± 0.00015). Assuming this procedure of data treatment, the percentage of error in the luminous transmittance of the filters is around 5%. The same mean value is given by MacAdam (see page 253 in reference [1]). According to MacAdam, the luminous transmittance of the filters is contained within an interval of ±10% with regard to the mean value. In order to individually analyze the difference for each filter with regard to the mean value, we calculate the percentage of deviation as $\Delta Y_f = \frac{\hat{Y}_f - Y_f}{\hat{Y}_f} 100$.

The results obtained are shown in Fig. 5. It should be pointed out that most of the filters present a luminous transmittance within an interval of 5% around the mean luminous transmittance and practically all lie in a band of 10%. Only the filters 78, 99, 110, 111 and 116 fall outside the band of 10%, although none of them differ more than 13%.

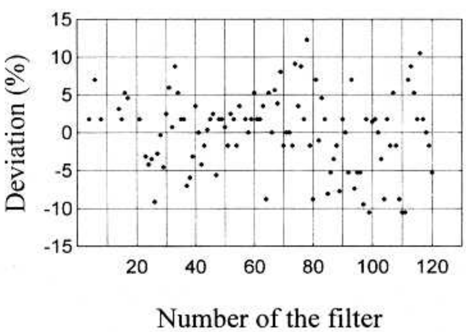

FIGURE 5. Percentage of deviation ΔY_f of luminous transmittance of each filter with regard to the average luminous transmittance. The X axis of the figure indicates the number of the filter given in Table I of reference [1].

TABLE III
CONTINUATION

Number of Int.	Series 1	Series 2	Chromaticity coordinates		d_I	e_{I1}	e_{I2}
			x_I	y_I			
27	7	9	0,145	0,667	0,0137	1,9	6,8
28	7	11	0,144	0,675	0,0073	1,0	2,0
29	7	13	0,144	0,677	0,0066	0,9	1,3
30	7	14	0,147	0,658	0,0219	3,0	8,8
31	9	11	0,152	0,669	0,0108	5,4	3,0
32	9	13	0,140	0,665	0,0175	8,7	3,6
33	9	14	0,176	0,677	0,0264	13,2	10,6
34	11	13	0,144	0,676	0,0075	2,1	1,5
35	11	14	0,158	0,665	0,0164	4,6	6,6
36	13	14	0,136	0,651	0,0319	6,5	12,8

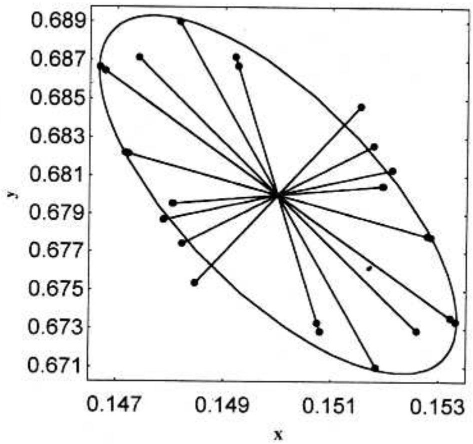

FIGURE 4. Distribution of points of equally noticeable chromaticity differences $(x_e, y_e)^s_m$ and $(x_e, y_e)^I_m$ obtained following MacAdam's procedure for color center 4. In the chromaticity diagram the straight lines represent the direction of chromaticity variation in each series of measurements. These lines are computed from the slopes provided by MacAdam. The contour of the ellipse corresponds to MacAdam's results and it is drawn according to the data supplied in reference [22].

TABLE III

Intersections among the straight lines generated in the different directions around the color center number 4. In the first column there appears the number assigned to the intersection. The second column contains the number of the series carried out on the first straight line of the intersection. The third column provides the number of the series carried out over the second straight line. The chromaticity coordinates of the intersection are listed in the fourth and fifth colums respectively. In the sixth column the distance from each intersection to MacAdam's center (x_m,y_m) is given. Finally, the quotiens between the distance d_I and the chromaticity standard deviations $(\Delta s_m)_1$ and $(\Delta s_m)_2$ given by MacAdam for the first and second straigh lines in the intersection are given in the seventh and eighth columns respectively.

Number of Int.	Series 1	Series 2	Chromaticity coordinates		d_I	e_{I1}	e_{I2}
			x_I	y_I			
1	1	3	0.180	0.549	0.1346	14,7	17,8
2	1	4	0,152	0,688	0,0081	0,9	2,6
3	1	5	0,157	0,662	0,0194	2,1	2,7
4	1	7	0,128	0,805	0,1265	13,8	17,6
5	1	9	0,156	0,671	0,0110	1,2	5,5
6	1	11	0,156	0,666	0,0152	1,7	4,2
7	1	13	0,150	0,696	0,0158	1,7	3,2
8	1	14	0,157	0,665	0,0165	1,8	6,6
9	3	4	0,138	0,667	0,0175	2,3	5,6
10	3	5	0,093	0,792	0,1255	16,6	17,5
11	3	7	0,149	0,635	0,0445	5,9	6,2
12	3	9	0,139	0,665	0,0188	2,5	9,4
13	3	11	0,132	0,685	0,0190	2,5	5,3
14	3	13	0,140	0,663	0,0200	2,6	4,1
15	3	14	0,142	0,656	0,0256	3,4	10,3
16	4	5	0,148	0,681	0,0027	0,9	0,4
17	4	7	0,144	0,676	0,0068	2,2	0,9
18	4	9	0,136	0,664	0,0216	6,9	10,8
19	4	11	0,144	0,676	0,0075	2,4	2,1
20	4	13	0,144	0,676	0,0076	2,4	1,5
21	4	14	0,121	0,642	0,0483	15,4	19,3
22	5	7	0,142	0,693	0,0148	2,1	2,1
23	5	9	0,153	0,670	0,0108	1,5	5,4
24	5	11	0,154	0,668	0,0130	1,8	3,6
25	5	13	0,147	0,684	0,0053	0,7	1,1
26	5	14	0,156	0,665	0,0167	2,3	6,7

The fact of obtaining two points, $(x_e,y_e)^s_m$ and $(x_e,y_e)^l_m$, for each series of measurements implies the *a priori* assumption that the visual sensitivity to small color differences is *symmetric* around a given color center in each direction. In other words, it means that equal-noticeability is independent of the way in which the color stimulus is varied in a fixed direction.

The data and results provided by MacAdam for each ellipse are as follows (see Table III, reference [1]):

- $f_2 \equiv$ number of the filter that provides the field chromaticity when $\theta=0^0$ (filter number 2),
- $f_1 \equiv$ number of the filter that provides the field chromaticity when $\theta=90^0$ (filter number 1),
- $\theta \equiv$ the mean value of the angle for each series of observations,
- x_m and y_m are the chromaticity coordinates of the color center,
- $\Delta\theta \equiv$ the standard deviation of the angle θ,
- $f_m \equiv$ the slope of the straight line generated by each pair of filters,
- $\Delta s_m \equiv$ standard deviation of the chromaticity with regard to the color center.

The slope f_m of each diameter of the ellipses was computed as the quotient of the standard deviations associated with the chromaticity coordinates of the color center, $f_m=\Delta y/\Delta x$.

We have calculated the chromaticity coordinates for the points $(x_e,y_e)^s_m$ and $(x_e,y_e)^l_m$ that configure MacAdam's ellipses using these data. The standard deviations of the chromaticity coordinates with regard to (x_m,y_m) for each series of measurements are related by the slope f_m as

$$\Delta y_m = f_m \Delta x_m \tag{7}$$

The standard deviation of the chromaticity is given by

$$\Delta(s_m)^2 = \Delta(x_m)^2 + \Delta\,(y_m)^2 \tag{8}$$

thus producing

$$\left|\Delta x_{\mathrm{m}}\right| = \sqrt{\frac{\Delta(s_{\mathrm{r}}}{1+f}} \tag{9}$$

After the calculation of $|\Delta x_m|$ we obtain $|\Delta y_m|$ from equation (7). Considering the sign of the slope f_m, the points $(x_e,y_e)^s_m$ and $(x_e,y_e)^l_m$ are obtained (see Fig. 3b)

$$(x_{\mathrm{e}},y_{\mathrm{e}})^S_{\mathrm{m}} = (x_{\mathrm{m}} + \frac{f_{\mathrm{m}}}{|f_{\mathrm{m}}|}\left|\Delta x_{\mathrm{m}}\right|, y_{\mathrm{m}} + \left|\Delta y_{\mathrm{m}}\right|)$$

and (10)

$$(x_{\mathrm{e}},\mathrm{y}_{\mathrm{e}})^l_{\mathrm{m}} = (x_{\mathrm{m}} - \frac{\partial_{\mathrm{m}}}{|\partial_{\mathrm{m}}|}\left|\Delta x_{\mathrm{m}}\right|, \mathrm{y}_{\mathrm{m}} - \left|\Delta y_{\mathrm{m}}\right|)$$

TABLE II

Distance d between the center (x_m,y_m) given by MacAdam and the point (x,y) representing the average chromaticity computed for each series of observations carried out in the different directions around the color center number 4 (second column). The third column shows the quotient between distance d and the corresponding standard deviation Δs_m given by MacAdam. The first column shows the number of each serie of observations.

Number of the series	**d**	**e**
1	0.0187	2.0
2	0.0194	2.1
3	0.0327	4.3
4	0.0234	7.5
5	0.0167	2.3
6	0.0075	1.0
7	0.0155	2.2
8	0.0067	1.0
9	0.0108	5.4
10	0.0118	5.8
11	0.0076	2.1
12	0.0072	2.0
13	0.0071	1.5
14	0.0209	8.4

filters to obtain one ellipse. Each pair of filters produced a different diameter of the ellipse (see Fig. 3a). According to MacAdam, each pair of filters was selected so that it could reproduce a chromaticity very close to that of the color center under consideration. Thus MacAdam implicitly assumed that, for a given color center, the different pairs of filters generated straight lines with a common intersection.

MacAdam had a filter capable of producing the chromaticity coordinates of the center of each of the 25 color centers. When comparing each pair of filters f_1 and f_2 with the aforementioned filter, he determined the angle θ_1 of the prism that produced the reference stimulus in the fixed half-field. The observer rotated the other prism until he considered that both fields were matched. The standard deviation of these observations was represented in the chromaticity diagram by two points $(x_e,y_e)^s{}_m$ and $(x_e,y_e)^l{}_m$ both of which lie on the straight line determined by the filters. These points are *symmetrically* situated on each side of the the color center at a distance Δs_m (see Fig. 3b). MacAdam computed the distance Δs_m from the mean value of the angle θ and its standard deviation $\Delta\theta$ using equation (6) as was explained in Section **3**. By means of repeating this process for each pair of filters, he obtains the points that generate the ellipse.

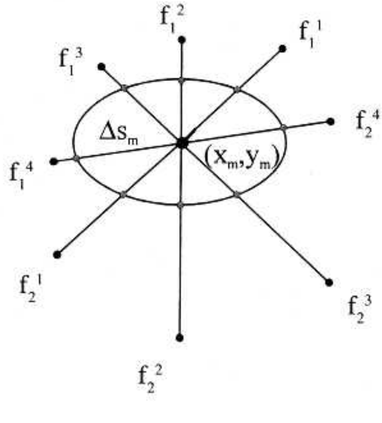

FIGURE 3(a).

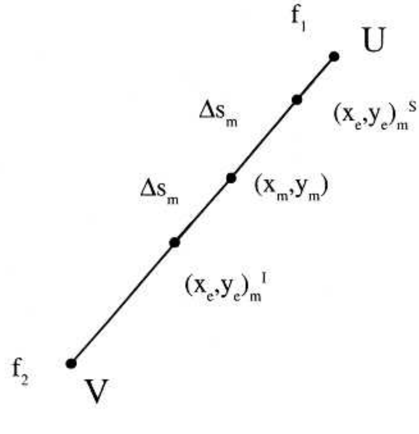

FIGURE 3(b).

FIGURE 3.

(a) Diameters of the ellipses generated by the different pairs of filters (solid points) used around a color center (x_m, y_m). The superindex assigned to each filter indicates the number of the series of color matchings carried out. Subindex 1 is assigned to the filter that provides the field chromaticity when the Rochon prism angle is $\theta=90^0$ and subindex 2 is assigned to the filter that provides the field chromaticity when the Rochon prism angle is $\theta=0^0$.

(b) Representation in the chromaticity diagram of the standard deviation of the observations carried out.

TABLE I

The number of the series of color matchings carried out in each direction around color center 4 are listed in the first column. This number is in accordance with the order given by MacAdam (Table III, Ref. 1). The calculated chromaticity coordinates (x,y) associated to the mean angle θ obtained in each direction aroud the color center 4 are listed in the second and third columns. The percentage of deviations of the coordinates given by MacAdam for the center of the ellipse, x_m and y_m, with regard to the computed coordinates appear in the two last columns.

Number of the series	**Chromaticity coordinates**		$\varepsilon(x_m)$ (%)	$\varepsilon(y_m)$ (%)
	x	**y**		
1	0.157	0.663	4.6	-2.6
2	0.157	0.662	4.6	-2.7
3	0.145	0.648	-3.4	-5.0
4	0.161	0.701	6.8	3.0
5	0.141	0.694	-6.1	2.1
6	0.152	0.673	1.3	-1.1
7	0.146	0.665	-3.0	-2.2
8	0.144	0.682	-4.5	0.3
9	0.153	0.670	2.2	-1.5
10	0.158	0.671	5.1	-1.3
11	0.144	0.676	-4.4	-0.6
12	0.147	0.674	-2.2	-0.9
13	0.147	0.687	-1.8	1.0
14	0.170	0.673	11.7	-1.0

MacAdam made the calculations from the experimental settings of the angle θ with the aid of a slide rule specifically designed for his investigation. The distance *UV* between the points representing the chromaticity coordinates of the filters employed in each series of equalizations will be designed by *S* and is given by

$$S \equiv UV = \sqrt{(x_u - x_v)^2 + (y_u - y_v)^2} \tag{4}$$

The chromaticity coordinates of the filters are easily calculated. The distance *VW* from the point representing the chromaticity of the equalization at an angle θ to the point representing the chromaticity of filter number 2 will be called *s* and is given by

$$s = VW = f(\theta)\ \mathrm{S} \tag{5}$$

MacAdam determined the distances *S* and *s* by measuring them on a large scale chromaticity diagram, and he computed the product given in equation (5) using the slide rule.

The chromaticity differences Δs corresponding to small changes $\Delta\theta$ in the Rochon prism reading are given by

$$\Delta \mathrm{s} = \frac{ds}{d\theta}\Delta\theta = \mathrm{S}\,\frac{df(\theta)}{d\theta}\Delta\theta \tag{6}$$

MacAdam computed the deviations Δs by subtracting the values of the function $f(\theta)$ given in the slide rule for values close to θ, dividing that result by the difference in the values of θ and multiplying it by the value of *S*. The value of Δθ was taken as the standard deviation of the mean value of the observed angles in 50 equalizations made by the observer in each direction around the same color center.

4. CHROMATICITY NOTICEABILITY CRITERION

MacAdam needed to establish a consistent criterion for the measurement of noticeable color differences. In previous investigations the criterion used was the just-noticeable difference. MacAdam proposed that the results of color equalization measurements should be given in terms of standard deviation, i.e., the mean square root of the individual measurements.

The mean value of the observed angles in the variable prism was considered the «correct» reading of the equalization. Using equation (6), the standard deviation of the prism angle, Δθ, is converted to standard deviation of distance, Δs, in the chromaticity diagram.

5. THE OBTAINING OF THE ELLIPSES

The ellipses obtained by MacAdam represent the standard deviations, in different directions, of color matching around the chromaticity given by the color center (x_m, y_m) (center of the ellipse). For a given ellipse, each direction in which chromaticity variation is studied, is generated by a different pair of filters, f_1 and f_2. He used from 5 to 8 pairs of the contribution of filter number 1 (horizontal polarization). Let X_v, Y_v and Z_v be the tristimulus values of the visual field when $\theta=0^0$, i.e., when the color of the field is only produced by the contribution of filter number 2 (vertical polarization). For any given angle of the Rochon prism, θ, the tristimulus values of the mixture of the stimulus coming from both filters will be given by

$$\begin{aligned} X &= X_u \sin^2\theta + X_v \cos^2\theta \\ Y &= Y_u \sin^2\theta + Y_v \cos^2\theta \\ Z &= Z_u \sin^2\theta + Z_v \cos^2\theta \end{aligned} \tag{2}$$

From equation (2) it is possible to compute the chromaticity coordinates that, in the CIE diagram, represent any color stimulus generated by the observer in the color equalization process. MacAdam did not used equation (2). In his words «... *The center of gravity principle provides a much more convenient method of locating the point...*» [1]. Let m_u be the «mass» assigned to filter number 1 ($m_u = X_u+Y_u+Z_u$) and let m_v be the «mass» assigned to filter number 2 ($m_v = X_v+Y_v+Z_v$). When the angle of the prism is θ, the point representing the mixture of both filters, divides the straight line joining the points that locate the filters in the chromaticity diagram in the relation $m_u\sin^2\theta$ to $m_v\cos^2\theta$. The chromaticity of the observation field when $\theta=90^0$ (filter f_1) corresponds to point U with chromaticity coordinates (x_u, y_u). The chromaticity of the observation field when $\theta=0^0$ (filter f_2) corresponds to point V with chromaticity coordinates (x_v, y_v). For a given generic angle θ, the chromaticity of the observation field corresponds to the point W *(x, y)* (see Fig. 2). The relation of distances *VW/VU* is represented by the fraction $f(\theta)$ defined as

$$f(\theta) = \frac{1}{r\cot^2\theta+1}, \tag{3}$$

and

$$r = \frac{m_v}{m_u}$$

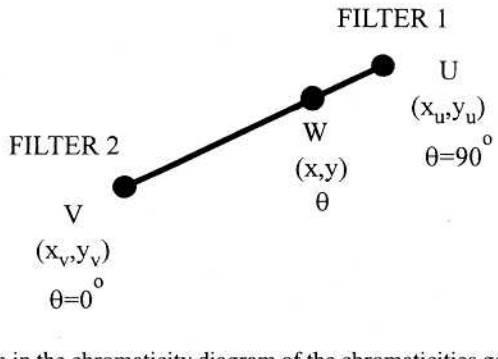

FIGURE 2. Representation in the chromaticity diagram of the chromaticities generated for the different Rochon prism angles.

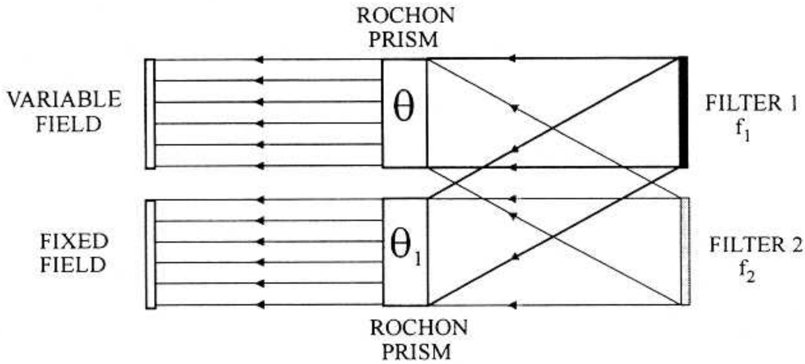

FIGURE 1. Diagram of MacAdam's apparatus.

field with the fixed half-field. The reading of the Rochon prism is 0^0 when, without deviation, the prism transmits polarized light in the vertical plane, whereas the reading is 90^0 when, without deviation, the prism transmits polarized light in the horizontal plane. Let U be the distribution of the energy coming from filter number 1 (polarized in the horizontal plane) that impinges on the Rochon prism, and let V be the distribution of the energy coming from filter number 2 (polarized in the vertical plane) that impinges on the Rochon prism. The total energy transmitted by the prism without deviation, W, for a generic value of the angle of the prism θ is given by

$$W = T(U \sin^2\theta + V \cos^2\theta), \tag{1}$$

T being the prism transmittance. If the values of the distributions of U and V are equal, then the value of the total transmitted distribution is constant, independently of the angle, and consequently, independently of the proportions in which the U and V distributions are combined. If the filters f_1 and f_2 have equal luminous transmittance for the light source considered, then the flux transmitted without deviation by the Rochon prism will be constant, independently of the angle θ and of the proportions in which the colours originated in the filters are combined. This characteristic guarantees the constant luminance on the observation field for any color stimulus generated. The apparatus used by MacAdam had two Rochon prisms (see Fig. 1). The color stimulus of the fixed field is obtained at a certain fixed angle θ_1. The stimulus that provides the equalization is generated by the observer when modifying angle θ of the second prism until he considers that both fields are «equal».

3. DETERMINATION OF CHROMATICITY DIFFERENCES OF A COLOR MATCHING: ΔS_M

The filters f_l and f_2 together with the angle of the first Rochon prism, θ_1, allow the specification of the fixed field color stimulus (see Fig. 1). With the same filters, the observer manipulates the control of the second Rochon prism until he obtains the equalization of the two fields for a certain angle, θ. Let X_u, Y_u and Z_u be the tristimulus values of the visual field when θ=90^0, i.e., when the color of the field is only produced by

The conclussions derived from MacAdam's study, i.e., the geometry and the size of the curves of equally noticeable chromaticity differences, have been questioned by some authors [2-11]. Nevertheless, MacAdam's ellipses and other results derived from them ([12-14]) are currently used for different purposes ([15-21]). The conclusions derived from our study may help to understand the discrepancies found between the results obtained in reference [1] and those obtained in other studies [2-11].

The results will be illustrated only for ellipse number 4 so as not to make this paper an excessively long one, although the calculations have been made for all the color centers considered by MacAdam. These results are available from the authors on request. We have selected this color center since it is the point with the largest number of matchings. The most outstanding results will be shown for the 25 color centers. In so far as it has been possible we have tried to maintain the original notation used by MacAdam.

In the following analysis we use the subindex *m* for all the magnitudes taken from the tables of reference [1] that were calculated by MacAdam, except for θ and $\Delta\theta$.

2. MEASUREMENT CONDITIONS AND SYNTHESIS OF THE EXPERIMENTAL APPARATUS

In this section we briefly describe the measurement conditions used in the investigation carried out by MacAdam and we resume the principle of operation of the ingenious apparatus developed for his investigation. All the observations were made at a constant luminance of 15 millilamberts. The comparison was made using a two degree comparison field, divided semicircularly. The periferal field substended an angle of 21 degrees when illuminated with C illuminant. The resultant luminance in the periferal field was of 7.5 millilamberts. A 2.6 mm artificial pupile was used to maintain the retinal illuminace at a fixed value. Color matchings were repeated between the variable stimulus and the fixed stimulus. Both fields presented the same luminance. The observer operated a unique control knob to produce color variations in the variable visual field until he considered that both fields were equal. This control was deliberately designed so that the positions (x, y) in the CIE 1931 chromaticity diagram of all the possible color matchings made by the observer, in all the series of measurements for each color center previously fixed, will lie in a straight line. The values of x and y cannot be selected independently of each other. In this sense, the color matchings are said to be «guided color matchings». All color matchings were «isomeric matchings»; in this way if the observer achieved an exact color equalization, the spectral distribution in each half-field was the same.

The color stimuli presented to the observer in the two halves of the field were obtained using different combinations in pairs of 105 filters. The filters were carefully selected and characterized, and their spectral transmittance was periodically verified. The filters were illuminated using a calibrated lamp with a color temperature of 2848 K (A illuminant).

The variation of the color stimuli in the field of observation is made using a Rochon prism. The light beams originate in two different filters and are perpendicularly polarized. The proportion of the beams originated in each filter that passes through the prism without deviation and impinges on the observation field depends on the prism azimuth θ. This angle is the parameter that the observer varies when trying to equalize the variable half-

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